Regularity Properties of Schrödinger and Dirichlet Semigroups

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First we compute Brownian motion expectations of some Kac's functionals. This allows a complete study of the semigroups generated by the formal differential operator $H = -\frac{1}{2}\Delta + V$ on the various Lebesgue's spaces $L^q = L^q\{\mathbb{R}^n, dx\}$, whenever the negative part of V is in $L^\infty + L^p$ for some $p > \max\{1, n/2\}$. Our approach is probabilistic and some of the proofs are surprisingly elementary. The negative infinitesimal generators of our semigroups are shown to be reasonable self-adjoint extensions of H. Under mild assumptions on V, H is unitary equivalent to the Dirichlet operator, say D, associated to its ground-state measure. We study regularity of the semigroups generated by D. We concentrate on hyper and supercontractivity and we give, using probabilistic techniques, new examples of potential functions V which give rise to hyper and supercontractive Dirichlet semigroups.

I. INTRODUCTION

There is a well known relation between the theory of diffusion processes, second order elliptic and parabolic equations and potential theory on the Euclidean space $\mathbb{R}^n$. In the present paper we take advantage of this interplay to study the imaginary time Schrödinger operator $-\frac{1}{2}\Delta + V$ in a manner which brings probabilistic mechanisms to the foreground.

This approach originates in the work of Kac [24]. Using Wiener's measure in the case of heat equation, he made mathematically rigorous the heuristic prescriptions given by R. Feynman to solve Schrödinger equation. It was a new momentum in this study and a wave of interest was initiated by this publication (see for example [31, 14, 2, 12, 6, 27, 1, 3, 5]). Although some of the results presented here are essentially known, our proofs are new, at least in nature. While these results agree with those found using differential equation techniques, as desired, it is informative to probe their anatomy in the path space picture more deeply.

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The relation of our results to certain problems in quantum field theory should not be unnoticed. Indeed, even though the discussion of this paper is confined solely to the study of particular quantum mechanical Hamiltonians, our work has been motivated in large measure, by a potential applicability of the concepts and methods we use to fundamental problems in quantum field theory.

Since the title may not give a clear idea of the scope of this article, we proceed to a summary of the contents of the following sections.

The first part of Section II covers the basic notations and definitions necessary to the following work. Since averages of various expressions with respect to Wiener measure are extensively used in this paper, we introduce the coordinate representation of Brownian motion Markov process, and we state and prove the properties of the so-called Kac's averages which we use in the present paper. There is none but a new result in this section (see Theorem 2.1 below). It provides us with a finer estimation than Berthier–Gaveau's one [3, Theorem 1]. Our proof draws its inspiration from Simon's proof of Berthier–Gaveau's theorem (see [43, Theorem II.2]) and Lieb's proof of a conjecture on the number of bound states for Schrödinger operator (oral communication). The crucial point of our estimate is the exhibition of a simple time dependence.

Section III is devoted to the definition and the study of semigroups generated by the formal differential operator $H = -\frac{1}{2}\Delta + V$ on various Lebesgue's spaces. These semigroups are defined via the so-called Feynman–Kac formula. This approach is not new (see for example [31, 14, 2, 27]), but, unfortunately, it required a severe restriction: the potential function V had to be assumed bounded below. Thanks to the above mentioned Kac's averages estimation, we have been able to push the proofs of [31, 14] in order to include unbounded below potential functions V. In fact the negative part of V is required to be in $L^{\infty}(\mathbb{R}^n, dx) + L^p(\mathbb{R}^n, dx)$ for some $p > \max\{1, n/2\}$. Among the smoothing properties we prove, let us single out the followings: (a) the semigroup generated by H maps boundedly $L^q(\mathbb{R}^n, dx)$ into $L^r(\mathbb{R}^n, dx)$ whatever the extended real numbers q and r are, provided q is finite and $q \leqslant r$, (b) this semigroup is strongly continuous on $L^q(\mathbb{R}^n, dx)$ if $q < +\infty$ and $V \in L^1_{loc}(\mathbb{R}^n, dx)$, (c) its range is contained in $C_0(\mathbb{R}^n)$ the space of continuous functions which vanish at infinity whenever $V \in L^{\bar{p}}_{loc}(\mathbb{R}^n, dx)$ for some $\bar{p} > n/2$. Similar regularity properties have been obtained quite recently by Herbst and Sloan [20]. Their conclusions are somewhat weaker in some places but they are also stronger in allowing more general local perturbations.

Most of the material of Section IV is essentially known. First we prove that the negative infinitesimal generator of Schrödinger semigroup acting in $L^q(\mathbb{R}^n, dx)$, say H_q, yields a reasonable self-adjoint extension of H. This generalizes some recent work of Semenov [37]. Second we check that H_2 coincides with H defined as sum of quadratic forms. Third we show that intricate proofs of some known results are streamlined by the methods of the probabilistic approach. Finally the section ends with a brief review on Dirichlet

forms associated to probability measures on $\mathbb{R}^n$, a technical lemma to exhibit a form core and a proof that H_2 is unitary equivalent to the Dirichlet operator associated to its groundstate measure.

Section V is devoted to the study of hyper and supercontractive properties of Dirichlet semigroups. The theory of hypercontractive semigroups is an abstraction of certain developments in quantum field theory (see e.g. [28, 17, 36, 34, 21, 18]). For the free Markov field, E. Nelson gave a final form and proved the best possible estimates [29]. Nelson's latest proof is probabilistic in nature. Among the numerous alternate proofs of his best hypercontractive estimates, we point out a very novel one [30] which involves sophisticated martingale representation theorems. Another elegant proof of Nelson's theorem was given by Gross [19] who proved that hypercontractive estimates are equivalent to logarithmic Sobolev inequalities; moreover he proved that the latters should be the correct substitute of classical Sobolev inequalities in the infinite dimensional setting because they provide semi-boundedness theorems for perturbed Hamiltonians in the same manner as classical Sobolev inequalities imply semiboundedness for some perturbations of the free Hamiltonian $-\frac{1}{2}\Delta$ on $L^2(\mathbb{R}^n, dx)$. Gross' very neat paper initiated the study of operator estimates via logarithmic Sobolev inequalities (see [13, 9, 11, 44, 33, 41]). Nevertheless means for proving these inequalities in the infinite dimensional setting of interacting quantum fields has not yet been found and it is hoped that the techniques presented in this paper will be of some help. The first part of Section V is nothing but a show of essentially known material. We state and prove equivalent forms of hypercontractivity for Dirichlet operators, one of them being a lower bound for the exponential fall off of the ground state eigenfunction in terms of the behavior at infinity of the potential function. This, together with results of [5, Section IV], provide us with non-spherically symmetric potential functions V which give rise to hypercontractive semigroups. These examples could not be known from [9, 33, 41]. The section ends with a detailed proof of a new example with a special emphasis on the key role of Brownian motion expectations. The main novelty of this section is certainly the use of stochastic process techniques in this area. We hope the present work serves to stimulate further research in this direction.

To keep the bibliography to a moderate length, we have adopted the convention that "see reference [×]" means "see reference [×] and the papers referred to therein."

II. Brownian Motion and Kac's Averages

In this paper $n \geqslant 1$ is a fixed integer. Our basic space is a real Hilbert space of dimension n (which we identify with $\mathbb{R}^n$), the inner product of which is denoted by a dot and the corresponding Hilbertian norm by $|\cdot|$.

First we introduce the notations we use for the various function spaces. $C(\mathbb{R}^n)$ denotes the space of continuous functions on $\mathbb{R}^n$; if $k \in [1, \infty]$ is an integer, we use the superscript k to denote the subspace of k-times differentiable functions and the subscript b (resp. 0, c) is used to specify that the functions are bounded (resp. tend to zero at infinity, have compact support). If $p \in [1, \infty]$ the symbols L^p will be used to denote the various Lebesgue spaces over $\mathbb{R}^n$ taken with Lebesgue's measure. As usual, we do not make any difference between functions and equivalence classes. $\| \cdot \|_p$ stands for the L^p-norm and $\| \cdot \|_{p,q}$ for the norm of an operator from L^p into L^q. When we use a measure μ instead of Lebesgue's measure, the dependence on the measure μ will be specified by $L^p(\mu)$ and $\| \cdot \|_{L^p(\mu)}$. For any subset U of a given set, $\mathbb{1}_U$ stands for the indicator function of U (i.e. $\mathbb{1}_U$ equals 1 on U and 0 outside), and for any real valued function f on $\mathbb{R}^n$, f_+ and f_- have the following meaning:

$$f_+ = f \mathbb{1}_{\{f \geqslant 0\}} \qquad \text{and} \qquad f_- = f_+ - f.$$

Second we fix the probabilistic notations and we review standard facts on Brownian motion which we are going to use. $\Omega = C(\mathbb{R}_+, \mathbb{R}^n)$ is the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^n$. For each $t \geqslant 0$

$$X_t: \Omega \ni \omega \mapsto X_t(\omega) = \omega(t)$$

is the tth coordinate function (for typing convenience X_t will sometimes be written $X(t)$),

$$\theta_t: \Omega \ni \omega \mapsto \theta_t \omega \in \Omega$$

is the time translation defined by:

$$[\theta_t \omega](s) = \omega(t + s) \qquad s \geqslant 0,$$

and $\mathscr{F}_t = \sigma\{X_s ; 0 \leqslant s \leqslant t\}$ is the smallest σ-field of subsets of Ω for which all the functions X_s with $0 \leqslant s \leqslant t$ are measurable. Now, if we set:

$$\mathscr{F} = \sigma\{X_t ; t \geqslant 0\},$$

$\mathscr{F}$ is nothing but the Borel σ-field of Ω equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. On the measurable space $(\Omega, \mathscr{F})$ there is a unique probability measure, say W, which satisfies:

(i) $W\{X_0 = 0\} = 0$.

(ii)   If $0 = t_0 < t_1 < \cdots < t_n$, on the probability space $(\Omega, \mathscr{F}, W)$, the random variables $X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independent, the jth being Gaussian with mean zero and variance $t_j - t_{j-1}$.

The construction of this measure is due to Wiener and can be found in any standard text book on stochastic processes (see for example [23, Chapter 1]

for several constructions). For each $x \in \mathbb{R}^n$ the probability measure W_x is defined by:

$$W_x\{A\} = W\{\tau_x(A)\} \qquad A \in \mathscr{F},$$

where τ_x is the path translation:

$$[\tau_x \omega](t) = x + \omega(t) \qquad x \in \mathbb{R}^n, \quad \omega \in \Omega, \quad t \geqslant 0.$$

The symbol E_{W_x} is used to denote the expectation and the conditional expectation with respect to W_x. The collection $\{W_x \, ; \, x \in \mathbb{R}^n\}$ is a Markov process in the following sense:

(i) $\forall x \in \mathbb{R}^n, \; W_x\{X_0 = x\} = 1$,

(ii) for any bounded $\mathscr{F}$-random variable Φ, the function $\mathbb{R}^n \ni x \mapsto E_{W_x}\{\Phi\}$ is Borel measurable.

(iii) for any $x \in \mathbb{R}^n$, $t > 0$, and any bounded $\mathscr{F}$-random variable Φ we have:

$$E_{W_x}\{\Phi \circ \theta_t \mid \mathscr{F}_t\} = E_{W_{X(t)}}\{\Phi\}. \tag{2.1}$$

Let us note that in the usual probabilistic terminology such a Markov process is called normal, time homogeneous and conservative. $\{W_x \, ; \, x \in \mathbb{R}^n\}$ is in fact a strong Markov process, that is, (2.1) is true, not only for constant times t, but for all $\mathscr{F}_t$-stopping times (this fact is due to Hunt [22]; see also [23, Sect. 1.6] for a proof). For each measurable function f on $\mathbb{R}^n$ we set:

$$[P_t f](x) = E_{W_x}\{f(X_t)\} \tag{2.2}$$

whenever this latter expression makes sense. Let us point out that:

$$[P_t f](x) = \int_{\mathbb{R}^n} f(y) \, p_t(x, y) \, dy$$

where $p_t(x, y)$ is the Brownian transition function, namely:

$$p_t(x, y) = (2\pi t)^{-n/2} \exp[-\mid x - y \mid^2 / 2t] \qquad t > 0, \quad x, y \in \mathbb{R}^n. \tag{2.3}$$

For each $p \in [1, \infty[$, $\{P_t \, ; \, t \geqslant 0\}$ is a strongly continuous contraction semigroup on L^p. If A_p denotes the negative infinitesimal generator we have:

$$\mathscr{D}(A_p) = \{f \in L^p; \, \Delta f \in L^p\} \qquad \text{and} \qquad A_p f = -\tfrac{1}{2} \Delta f \qquad \text{if} \quad f \in \mathscr{D}(A_p) \tag{2.4}$$

where $\mathscr{D}(A)$ stands for the domain of the operator A and where Δf means Laplacian of f in the sense of distributions. Furthermore, for each $t > 0$ and each $p \in [1, \infty[$, P_t is a bounded operator from L^p into L^∞ whose norm satisfies:

$$\| P_t \|_{p, \infty} \leqslant c(p) \, t^{-n/2p} \tag{2.5}$$

where:

$$c(p) = \begin{cases} (2\pi)^{-n/2} & \text{if } p = 1 \\ (2\pi)^{-n/2p}(1 - p^{-1})^{(1-p^{-1})n/2} & \text{if } 1 < p < \infty \end{cases} \tag{2.6}$$

Now, if $p \geqslant 1$, $p > n/2$, $t > 0$, and $f \in L^p$, (2.2) and (2.5) imply:

$$\sup_{x \in \mathbb{R}^n} \int_0^t E_{W_x}\{f(X_s)\} \, ds \leqslant (1 - n/2p)^{-1}t^{1-n/2p}c(p) \|f\|_p . \tag{2.7}$$

In order to state the only new result of this section we have to recall the definition of the so-called Mittag–Leffler functions:

$$e_\epsilon(x) = \sum_{k \geqslant 0} \frac{x^k}{\Gamma(1 + k\epsilon)} \qquad x \in \mathbb{R}, \quad \epsilon > 0. \tag{2.8}$$

THEOREM 2.1. *If V is a positive function in L^p for some $p > n/2$, then for each $t > 0$ we have*:

$$\sup_{x \in \mathbb{R}^n} E_{W_x}\left\{\exp\left[\int_0^t V(X_s) \, ds\right]\right\} \leqslant e_\epsilon(c(p) \| V \|_p \Gamma(\epsilon) t^\epsilon) \tag{2.9}$$

where $\epsilon = 1 - n/2p$.

Proof. If $k \geqslant 1$ is any integer and if $0 < s_1 < \cdots < s_k < t$ we have:

$$E_{W_x}\{V(X(s_1)) \cdots V(X(s_k))\}$$
$$= E_{W_x}\{V(X(s_1)) \cdots V(X(s_{k-1})) E_{W_x}\{V(X(s_k)) \mid \mathscr{F}_{s_{k-1}}\}\}$$
$$= E_{W_x}\{V(X(s_1)) \cdots V(X(s_{k-1})) E_{W_{X(s_{k-1})}}\{V(X(s_k - s_{k-1}))\}\}$$
$$\leqslant c(p) \| V \|_p (s_k - s_{k-1})^{-n/2p} E_{W_x}\{V(X(s_1)) \cdots V(X(s_{k-1}))\}$$

where we used the Markov property (2.1) and (2.5). A k steps induction gives:

$$E_{W_x}\{V(X(s_1)) \cdots V(X(s_k))\}$$
$$\leqslant c(p)^k \| V \|_p^k s_1^{-n/2p}(s_2 - s_1)^{-n/2p} \cdots (s_k - s_{k-1})^{-n/2p}, \tag{2.10}$$

which yields:

$$E_{W_x}\left\{\left(\int_0^t V(X_s) \, ds\right)^k\right\} = k! \int_0^t \int_{s_1}^t \cdots \int_{s_{k-1}}^t E_{W_x}\{V(X(s_1)) \cdots V(X(s_k))\} \, ds_1 \cdots ds_k$$
$$\leqslant k! \, (c(p) \| V \|_p \Gamma(\epsilon) t^\epsilon)^k / \Gamma(1 + k\epsilon) \tag{2.11}$$

by integrating the right hand side of (2.10). Now, in order to prove (2.9) it suffices to expand the exponential, to interchange the expectation and the expansion, (2.11) and the definition of ϵ and (2.8). ∎

The idea behind the proof of Theorem 2.1 is not new. It has been taken from Simon's proof of Berthier–Gaveau's theorem and Lieb's proof of a conjecture on the number of bound states for Schrödinger operator (oral communication. See also [26]). We refer to [3, Theorem 1] for Berthier–Gaveau's theorem, to [43, Theorem 11.2] for Simon's proof of this theorem and to the section on bound states problems of [43] for Lieb's result.

The crucial point of our result is the estimation of the dependence on t for Kac's averages. This is possible thanks to the following standard property of Mittag–Leffler functions:

$$\lim_{x\to\infty} e_\epsilon(x) - \epsilon^{-1} \exp[x^{1/\epsilon}] = 0 \tag{2.12}$$

For later reference we state here a standard property of Brownian motion expectations which we use in different places. For each $t > 0$, the path space transformation r_t is defined by:

$$[r_t\omega](s) = \begin{cases} \omega(t-s) & \text{if } 0 \leqslant s \leqslant t \\ \omega(0) & \text{if } s > t. \end{cases}$$

Then for any positive $\mathscr{F}_t$-random variable Φ we have:

$$\int_{\mathbb{R}^n} E_{W_x}\{\Phi\} \, dx = \int_{\mathbb{R}^n} E_{W_x}\{\Phi \circ r_t\} \, dx \tag{2.13}$$

III. HEAT SEMIGROUP IN A POTENTIAL OF CLASS $\mathscr{V}$

First we define the class of potential functions which we work with.

DEFINITION 3.1. An extended real valued function V on $\mathbb{R}^n$ is said of class $\mathscr{V}$ (or belonging to $\mathscr{V}$) if V is measurable and if $V_- \in L^\infty + L^p$ for some $p \geqslant 1$ which satisfies $p > n/2$ (such a p will sometimes be denoted $p(V)$).

Whenever we consider a breakup $V = V_1 - V_2$ of a potential of class $\mathscr{V}$ it will be implicitly assumed that V_1 is measurable and bounded below, $V_2 \geqslant 0$ and $V_2 \in L^{p(V)}$. Now, if $V \in \mathscr{V}$, if $t > 0$ and if $x \in \mathbb{R}^n$, by (2.7) we have:

$$0 \leqslant \int_0^t V_2(X_s) \, ds < +\infty \qquad W_x\text{-a.s.}$$

and thus:

$$-\infty < \int_0^t V(X_s) \, ds \leqslant +\infty \qquad W_x\text{-a.s.}$$

As we use the convention $e^{-\infty} = 0$ Kac's averages are meaningfull for potentials of class $\mathscr{V}$. Theorem 2.1 implies that for each $t > 0$ and $r > 0$:

$$K(r, t) = \sup_{x \in \mathbb{R}^n} E_{W_x} \left[\exp \left(-r \int_0^t V(X_s)\, ds \right) \right] < \infty. \tag{3.1}$$

Remark 3.1.   As it is easily seen, using (2.9) and (2.12) we have:

$$K(r, t) \leqslant k_\epsilon \exp[(c(p))^{1/\epsilon} \| V_2 \|_p^{1/\epsilon} \Gamma(\epsilon)^{1/\epsilon} r^{1/\epsilon} - r \inf V_1) t] \tag{3.2}$$

where $\epsilon = 1 - n/2p$ and where the dependence of the constant k_ϵ on ϵ could be precised.

Now let us define the heat semigroup via the so-called Feyman–Kac's formula:

DEFINITION 3.2.   If V is a potential function of class $\mathscr{V}$, if f is a measurable function on $\mathbb{R}^n$, if $x \in \mathbb{R}^n$ and if $t > 0$ we set:

$$[T_t^V f](x) = E_{W_x} \left\{ f(X_t) \exp \left[-\int_0^t V(X_s)\, ds \right] \right\} \tag{3.3}$$

whenever this expression makes sense.

For typing convenience we omit the superscript V whenever the dependence on the potential function is clear and no misunderstanding possible.

Before concentrating on Lebesgue's spaces let us study, for a short while, these operators on larger classes of functions. Following [14, Sect. 3] we set:

DEFINITION 3.3.   A Borel measurable function f on $\mathbb{R}^n$ is said to be moderate if for each $a > 0$ we have:

$$\int_{\mathbb{R}^n} |f(x)| e^{-a|x|^2}\, dx < \infty.$$

Let $\mathscr{M}$ be the set of moderate functions. It is a translation invariant linear space contained in L^1_{loc} and which contains all the L^p for $p \in [1, \infty]$. If $f \in \mathscr{M}$ and if $a > 0$ we have:

$$\int_{\mathbb{R}^n} |[T_t f](x)|\, e^{-a|x|^2}\, dx$$

$$\leqslant \int_{\mathbb{R}^n} E_{W_x} \left\{ |f(X_t)| \exp \left[-\int_0^t V(X_s)\, ds \right] \right\} e^{-a|x|^2}\, dx$$

$$= \int_{\mathbb{R}^n} |f(x)|\, E_{W_x} \left\{ e^{-a|X_t|^2} \exp \left[-\int_0^t V(X_s)\, ds \right] \right\} dx$$

$$\leqslant K(2, t)^{1/2} (4at + 1)^{-n/2} \int_{\mathbb{R}^n} |f(x)| \exp[-|x|^2/2(t + 1/4a)]\, dx$$

$$< \infty, \tag{3.4}$$

where we used (2.13), Hölder's inequally and the fact that for each $\sigma > 0$ we have:

$$E_{W_x}\{e^{-|X_s|^2, 2\sigma}\} = [\sigma_/(\sigma + s)]^{n/2} e^{-|x|^2/2(\sigma+s)}$$

In fact (3.4) proves that (3.3) makes sense for a.e. $x \in \mathbb{R}^n$ and that the function so defined is moderate whenever f is moderate. If moreover $|f|^q$ is assumed to be moderate for some $q > 1$, then (3.3) makes sense for all $x \in \mathbb{R}^n$ since:

$$E_{W_x}\left\{f(X_t)\exp\left[-\int_0^t V(X_s)\,ds\right]\right\} \leqslant E_{W_x}\{|f(X_t)|^q\}^{1/q}K(q', t)^{1 \cdot q'}$$

where q' denotes the conjugate exponent of q (i.e. $q^{-1} + q'^{-1} = 1$). From (3.3) it is easily seen that:

$$U \leqslant V \text{ a.e. } \Rightarrow T_t^V f \leqslant T_t^U f \text{ a.e.}$$

whenever U and V are potential functions of class $\mathscr{V}$, $t > 0$ and f is a non-negative moderate function. Moreover, from (2.7) it is clear that:

$$U = V \text{ a.e. } \Rightarrow T_t^V f = T_t^U f \text{ a.e.}$$

Later on in this paper (see Proposition 4.1 below) we will infer properties on T_t^V from corresponding, properties of operators $T_t^{V_k}$ where the V_k are truncated potentials that approximate V. The very approximation argument is based on the following: if $\{V_k ; k \geqslant 1\}$ is a nondecreasing (resp. nonincreasing) sequence of potentials of class $\mathscr{V}$ such that:

$$\lim_{k \to \infty} V_k = V \quad \text{a.e.}$$

for some $V \in \mathscr{V}$ (resp. and such that moreover:

$$V_k = V_+ - U_k \qquad k = 1, 2,\ldots$$

where the U_k are uniformly bounded below), then, for each $t > 0$, we have:

$$\lim_{k \to \infty} [T_t^{V_k} f](x) = [T_t^V f](x) \tag{3.5}$$

for almost every $x \in \mathbb{R}^n$ if $f \in \mathscr{M}$ and every $x \in \mathbb{R}^n$ if $|f|^q \in \mathscr{M}$ for some $q > 1$; furthermore, if $f \in L^q$ for some q such that $1 \leqslant q < +\infty$ the limit in (3.5) takes place in L^q-sense. Proofs consist of straightforward uses of monotone and dominated convergence theorems.

Using Brownian Markov property, it is easy to prove:

$$[T_{s+t}f](x) = [T_t(T_s f)](x) \tag{3.6}$$

for almost every $x \in \mathbb{R}^n$ if $f \in \mathcal{M}$ and every $x \in \mathbb{R}^n$ if $|f|^q \in \mathcal{M}$ for some $q > 1$. For example, the proof given in [14, Theorem 3.1] in the case $V \geqslant 0$, works without any change, in the general case of potentials of class $\mathscr{V}$.

Next we concentrate on the semigroup $\{T_t \, ; \, t \geqslant 0\}$ acting on the various Lebesgue spaces.

PROPOSITION 3.1.   *Let V be a potential function of class $\mathscr{V}$, let $q \in [1, \infty]$ and $t > 0$. Then T_t is a bounded operator on L^q and its norm satisfies:*

$$\| T_t \|_{q,q} \leqslant K(1, t). \tag{3.7}$$

If q' denotes the conjugate exponent of q, if $f \in L^q$ and if $g \in L^{q'}$ we have:

$$\int_{\mathbb{R}^n} [T_t f](x) \, \overline{g(x)} \, dx = \int_{\mathbb{R}^n} f(x) \overline{[T_t g](x)} \, dx. \tag{3.8}$$

Furthermore T_t is a bounded operator from L^q into L^r for all extended real numbers q and r in $[1, \infty]$ provided q is finite and $q \leqslant r$. Finally, if $f \in L^q$ for some finite $q \geqslant 1$ we have:

$$\lim_{|x| \to \infty} [T_t f](x) = 0 \tag{3.9}$$

Proof.   From the definitions of T_t and of $K(1, t)$ it is clear that (3.7) is satisfied when $q = \infty$. Furthermore if $f \in L^1$ we have:

$$\| T_t f \|_1 \leqslant \int_{\mathbb{R}^n} E_{W_x} \left\{ |f(X_t)| \exp\left[-\int_0^t V(X_s) \, ds \right] \right\} dx$$

$$= \int_{\mathbb{R}^n} |f(x)| \, E_{W_x} \left\{ \exp\left[-\int_0^t V(X_s) \, ds \right] \right\} dx$$

$$\leqslant K(1, t) \| f \|_1 \, ,$$

where we used (2.13). This proves (3.7) when $q = 1$, and the general case follows Riesz–Thorin interpolation theorem. If f and g are assumed to be nonnegative (3.8) is a straightforward consequence of (2.13). Indeed:

$$\int_{\mathbb{R}^n} [T_t f](x) \, g(x) \, dx = \int_{\mathbb{R}^n} E_{W_x} \left\{ f(X_t) \, g(X_0) \exp\left[-\int_0^t V(X_s) \, ds \right] \right\} dx$$

$$= \int_{\mathbb{R}^n} E_{W_x} \left\{ f(X_0) \, g(X_t) \exp\left[-\int_0^t V(X_s) \, ds \right] \right\} dx$$

$$= \int_{\mathbb{R}^n} f(x)[T_t g](x) \, dx,$$

and the extension to general f and g is obvious. Now if $1 < q < \infty$ and if q' is the conjugate exponent of q, for any $x \in \mathbb{R}^n$ we have:

$$
\begin{aligned}
|[T_t f](x)| &\leqslant E_{W_x}\{|f(X_t)|^q\}^{1/q} E_{W_x} \left\{ \exp\left[-q' \int_0^t V(X_s)\, ds\right] \right\}^{1/q'} \\
&\leqslant [P_t\, |f|^q](x)^{1/q} K(q', t)^{1/q'} \\
&\leqslant \|P_t\|_{1,\infty}^{1/q} \| \, |f|^q \|_1^{1/q} K(q', t)^{1/q'} \\
&\leqslant (2\pi)^{-n/2q} t^{-n/2q} K(q', t)^{1/q'} \|f\|_q,
\end{aligned}
\tag{3.10}
$$

which proves that:

$$
\| T_t \|_{q,\infty} \leqslant (2\pi)^{-n/2q}\, K(q', t)^{1/q'}\, t^{-n/2q}.
$$

Now (3.9) follows (3.10) and the corresponding property of the semigroup $\{P_t \, ; \, t \geqslant 0\}$. Consequently the last assertion of Proposition 3.1 is proved when $q > 1$ and $r = \infty$. In order to capture the others cases we use the semigroup property (3.6) and the self-adjointness property (3.8). ∎

Remark 3.2. The finiteness of $\| T_t \|_{q,\infty}$ and (3.9) are uniform in t restricted to any interval $[t_0, t_1]$ such that $0 < t_0 < t_1 < \infty$ and uniform in V restricted to a subclass of $\mathscr{V}$ in which a breakup $V = V_1 - V_2$ can be found with the V_1 uniformly bounded below and the $\| V_2 \|_p$ bounded for one $p > n/2$ common to all the V.

Remark 3.3. Formulae virtually identical to (3.10) had already been used in [6] and [20]. Herbst and Sloan's results in [20] are similar to some of ours.
   They are stronger in allowing more general local perturbations but weaker in proving boundedness of T_t only from L^q to L^r for q and r restricted to suitable intervals.
   In order to prove that the semigroup $\{T_t \, ; \, t \geqslant 0\}$ is strongly continuous on the various Lebesgue spaces we need the following:

LEMMA 3.1. *If $V \in L^1_{\text{loc}}$ is nonnegative, for almost every $x \in \mathbb{R}^n$ we have:*

$$
W_x \left\{ \lim_{t \to 0} \int_0^t V(X_s)\, ds = 0 \right\} = 1
\tag{3.11}
$$

Proof. Let us assume first that $V \in L^1$ and let $t > 0$. By Fubini's theorem we have:

$$
\int_{\mathbb{R}^n} E_{W_x} \left\{ \int_0^t V(X_s)\, ds \right\} dx = \int_0^t \int_{\mathbb{R}^n} E_{W_x}\{V(X_s)\}\, dx\, ds = t \| V \|_1.
$$

Now, using twice Fubini's theorem we obtain:

$$\int_0^t V(X_s)\,ds < \infty \qquad W_x\text{-a.s.}$$

for almost every $x \in \mathbb{R}^n$, and this yields (3.11). Now, let us assume that $V \in L^1_{loc}$ is nonnegative. For each integer $k \geqslant 1$ let us set:

$$V_k = V\mathbf{1}_{\{|y|\leqslant k\}}.$$

$V_k \in L^1$ and thus we can choose a negligeable subset N_k of $\mathbb{R}^n$ such that:

$$x \notin N_k \Rightarrow W_x \left\{\lim_{t\to 0}\int_0^t V_k(X_s)\,ds = 0\right\} = 1.$$

Now, if $x \notin N = \bigcup_k N_k$ is fixed and if k is such that $k > |x|$, W_x-almost surely we can choose a $t > 0$ such that:

$$\int_0^t V_k(X_s)\,ds < \infty. \tag{3.12}$$

Now, as V and V_k coincide in an open neighbourhood of x, decreasing t if necessary, (3.12) remains true if we replace V_k by V. This concludes the proof. ∎

PROPOSITION 3.2. *If V is a potential function of class $\mathscr{C}$ such that $V_+ \in L^1_{loc}$, then for any $q \in [1, \infty[$, $\{T_t \,;\, t \geqslant 0\}$ is a strongly continuous semigroup on L^q.*

Proof. $C_c(\mathbb{R}^n)$ being dense in L^q and $\|T_t\|_{q,q}$ being uniformly bounded in t restricted to any bounded neighbourhood of zero, it suffices to prove:

$$\lim_{t\to 0} T_t f = f$$

in L^q-sense for all $f \in C_c(\mathbb{R}^n)$. Moreover, without any loss of generality we may assume $f \geqslant 0$. By Lemma 3.1, there is a Borel subset of $\mathbb{R}^n$, say N, whose Lebesgue's measure is zero and which satisfies:

$$x \notin N \Rightarrow \left(\lim_{t\to 0}\int_0^t V(X_s)\,ds = 0 \quad W_x\text{-a.s.}\right)$$

Since f is continuous, if $x \notin N$ we have:

$$\lim_{t\to 0} f(X_t)\exp\left[-\int_0^t V(X_s)\,ds\right] = f(x) \qquad W_x\text{-a.s.} \tag{3.13}$$

Moreover, if $r > 1$ we have:

$$\sup_{0<t\leqslant 1} E_{W_x}\left\{f(X_t)^r \exp\left[-r\int_0^t V(X_s)\,ds\right]\right\} \leqslant \|f\|_\infty^r \sup_{0<t\leqslant 1} K(r,t) < \infty,$$

which proves that the net $\{f(X_t)\exp[-\int_0^t V(X_s)\,ds]; 0 < t \leqslant 1\}$ is equi-integrable near zero, and since, by (3.13), it converges W_x-a.s., we conclude that it converges in $L^1(\Omega, W_x)$, and consequently we have:

$$x \notin N \Rightarrow \lim_{t\to 0}[T_t f](x) = f(x). \tag{3.14}$$

The above argument did not use the compactness of the support of f, and thus, (3.14) is true for $f = 1_{\mathbb{R}^n}$. Namely we have:

$$x \notin N \Rightarrow \lim_{t\to 0} E_{W_x}\left\{\exp\left[-\int_0^t V(X_s)\,ds\right]\right\} = 1,$$

which, together with:

$$\sup_{0 < t \leqslant 1} K(1, t) < +\infty$$

implies, by the dominated converge theorem that:

$$\lim_{t\to 0}\| T_t f \|_q = \| f \|_q \tag{3.15}$$

for $q = 1$. (3.15) remains true if $q > 1$ because the bound needed to use Lebesgue's theorem is provided by (3.10). Now, the conjunction of (3.14) and (3.15) yields the desired conclusion. ∎

Remark 3.4. By the self-adjointness property (3.8) it follows that $\{T_t\,;\,t \geqslant 0\}$ is continuous on L^∞ for the weak-*-topology, but, as we will see later on (see Proposition 3.3 below) at least under mild conditions on V, the semigroup cannot be strongly continuous on L^∞.

Remark 3.5. When $V \geqslant 0$, the technical assumption $V \in L^1_{loc}$ can be dropped and replaced by the conclusion of Lemma 3.1. In [14] and [27] this remark is further discussed.

For each $q \in [1, \infty[$ let us denote by $-H_q$ the infinitesimal generator of the semigroup $\{T_t\,;\,t \geqslant 0\}$ on the Banach space L^q. By (3.7) and (3.2) we have:

$$\forall t \geqslant 0 \qquad \| T_t \|_{q,q} \leqslant Ke^{tE} \tag{3.16}$$

for some positive constants K and E independent of t. In the case $q = 2$, H_2 is symmetric because the T_t are self-adjoint. Moreover (3.16) implies that H_2 is self-adjoint and bounded below by $-E$. In fact, (3.16) says that $]-\infty, -E[$ is contained in the resolvent set of H_q whatever $q \geqslant 1$ is.

The study of these infinitesimal generators is postponed to next section. We prove now that nice regularity properties of the operators T_t can be obtained via mild restrictions on the potential function V.

PROPOSITION 3.3. *If V is a potential function of class $\mathscr{V}$, if $t > 0$ and if $f \in L^q$ for some $q \in [1, \infty]$ then $T_t f$ is continuous on the open subset of $\mathbb{R}^n$ whose points are those x which possess a neighborhood on which the $p(x)$th power of V_+ is integrable for some $p(x) > n/2$; in particular, $T_t f$ is continuous on $\mathbb{R}^n$ if $V_+ \in L_{loc}^{+n/2}$. (A measurable function g belongs to $L_{loc}^{+\alpha}$ if for any compact set K in $\mathbb{R}^n$ there is a real number $p(K) > \alpha$ such that $g \mathbb{1}_K \in L^{p(K)}$).*

Proof. Because of the smoothing properties of the semigroup $\{P_t \; ; \; t \geqslant 0\}$, Proposition 3.3 is an immediate consequence of the following:

LEMMA 3.2. *Let K be a compact subset of $\mathbb{R}^n$, U be a bounded open neighbourhood of K and $V = V_1 - V_2$ a potential function of class $\mathscr{V}$ such that $V_1 \mathbb{1}_U \in L^{\bar{p}}$ for some $\bar{p} > n/2$. Then, for each $\epsilon > 0$, for each interval $[t_0, t_1]$ such that $0 < t_0 < t_1 < \infty$, and for each $q \in [1, \infty]$ we can choose a positive number r_0 such that for any $f \in L^q$ we have:*

$$\sup_{0 < r < r_0} \sup_{t_0 \leqslant t \leqslant t_1} \sup_{x \in K} |[T_t f](x) - [P_r(T_{t-r} f)](x)| < \epsilon \|f\|_q \qquad (3.17)$$

Proof. By Remark 3.2 and the semigroup property it suffices to prove (3.17) when $q = \infty$. Let ϵ, t_0 and t_1 be fixed and let $\gamma > 0$ be such that:

$$\gamma \sup_{0 \leqslant t \leqslant t_1 - t_0/2} K(1, t) < \epsilon/2. \qquad (3.18)$$

Once such a $\gamma > 0$ is fixed, let us choose $a > 0$ such that for any real number b we have:

$$|b| < a \Rightarrow |e^{-b} - 1| < \gamma. \qquad (3.19)$$

Now let us assume that r is such that $0 < r < t_0/2$ and for such an r let us set:

$$A_r = \left\{ \left| \int_0^r V(X_s) \, ds \right| < a \right\}. \qquad (3.20)$$

Finally we fix $f \in L^\infty$ and $x \in \mathbb{R}^n$. By the Markov property of Brownian motion we have:

$$[T_t f](x) = E_{W_x} \left\{ f(X_t) \exp \left[-\int_r^t V(X_s) \, ds \right] \right\}$$

$$+ E_{W_x} \left\{ f(X_t) \left(\exp \left[-\int_0^r V(X_s) \, ds \right] - 1 \right) \exp \left[-\int_r^t V(X_s) \, ds \right] \right\}$$

$$= [P_r(T_{t-r} f)](x)$$

$$+ E_{W_x} \left\{ f(X_t) \left(\exp \left[-\int_0^r V(X_s) \, ds \right] - 1 \right) \exp \left[-\int_r^t V(X_s) \, ds \right]; A_r \right\}$$

$$+ E_{W_x} \left\{ f(X_t) \left(\exp \left[-\int_0^r V(X_s) \, ds \right] - 1 \right) \exp \left[-\int_r^t V(X_s) \, ds \right]; A_r^c \right\}$$

$$= [P_r(T_{t-r} f)](x) + (i) + (ii) \qquad (3.21)$$

But, because of (3.20) and (3.19) we have:

$$|(i)| \leqslant \gamma E_{W_x} \left\{ |f(X_t)| \exp\left[-\int_r^t V(X_s)\,ds\right] \right\}$$

$$= \gamma[P_r(T_{t-r}\,|f|)](x)$$

$$\leqslant \|f\|_\infty \, \epsilon/2 \tag{3.22}$$

where we used once more the Markov property of Brownian motion, (3.7) with $q = \infty$ and (3.18). Moreover we have:

$$|(ii)| \leqslant E_{W_x} \left\{ |f(X_t)| \exp\left[-\int_0^t V(X_s)\,ds\right]; A_r{}^c \right\}$$

$$+ E_{W_x} \left\{ |f(X_t)| \exp\left[-\int_r^t V(X_s)\,ds\right]; A_r{}^c \right\}$$

$$= (iii) + (iv), \tag{3.23}$$

with:

$$(iii) \leqslant \|f\|_\infty \sup_{0\leqslant t\leqslant t_1} K(2,t)^{1/2} W_x\{A_r{}^c\}^{1/2}, \tag{3.24}$$

and:

$$(iv) \leqslant E_{W_x} \left\{ |f(X_t)|^2 \exp\left[-2\int_r^t V(X_s)\,ds\right] \right\}^{1/2} W_x\{A_r{}^c\}^{1/2}$$

$$= [P_r(T_{t-r}^{2V}\,|f|^2)](x)^{1/2} W_x\{A_r{}^c\}^{1/2}$$

$$\leqslant \|f\|_\infty \sup_{0\leqslant t\leqslant t_1 - t_0/2} K(2,t)^{1/2} W_x\{A_r{}^c\}^{1/2}. \tag{3.25}$$

By (3.21)–(3.25) we will conclude when we prove:

$$\lim_{r_0\to 0} \sup_{0<r<r_0} \sup_{x\in K} W_x\{A_r{}^c\} = 0. \tag{3.26}$$

Now, let d be the distance from K to the complementary set U^c of U, and for each $s > 0$ let us set:

$$B_s = \{X_u \in U;\ 0 \leqslant u \leqslant s\}.$$

For any given $\delta > 0$ we can choose an $s > 0$ such that:

$$\sup_{x\in K} W_x\{B_s{}^c\} < \delta/2. \tag{3.27}$$

Indeed standard text books tell us that:

$$\lim_{s\to 0} W\{\sup_{0\leqslant u\leqslant s} |X_u| > d\} = 0.$$

(see [31, Lemma 2] for a proof; one can also obtain a good estimate of the above probability from Levy's maximal inequality, see for example [5, Formulae 2.4]). Moreover, for such an s, and for all $x \in K$ we have:

$$W_x\{B_s \cap A_r{}^r\} \leqslant W_x \left\{ \left| \int_0^r (V \mathbb{1}_U)(X_u) \, du \right| > a \right\}$$

$$\leqslant a^{-1} \int_0^r [P_u(|V|\, \mathbb{1}_U)](x) \, du$$

$$\leqslant a^{-1} \| V \mathbb{1}_U \|_{\bar{p}} c(\bar{p})(1 - n/2\bar{p})^{-1} r^{1-n/2\bar{p}},$$

with can be made less than $\delta/2$ provided r is small enough; this, together with (3.27) proves (3.26). ▮

Remark 3.6. In the case V_+ is bounded one can say more about the regularity of $T_t f$ for f in some of the L^q. Indeed, from [20, Theorem 9.7] it follows that $T_t f$ is Hölder continuous.

LEMMA 3.3. *Let V be a real valued, bounded below, Borel measurable function on $\mathbb{R}^n$ which satisfies:*

$$\lim_{|x| \to \infty} V(x) = +\infty. \tag{3.28}$$

Then, for each $t > 0$ we have:

$$\lim_{a \to \infty} \sup_{x \in \mathbb{R}^n} E_{W_x} \left\{ \exp\left[-\int_0^t V(X_s) \, ds \right]; \, |X_t| > a \right\} = 0. \tag{3.29}$$

Proof. If $\epsilon > 0$ is fixed, standard properties of Brownian paths (again see, for example [31, Lemma 2] or the same [5, Formulae 2.4]) implies the existence of a $d > 0$ such that:

$$\exp[t \mid \inf V \mid] W_0\{\sup_{0 \leqslant s \leqslant t} |X_s| > d\} < \epsilon/2. \tag{3.30}$$

Once such a $d > 0$ is fixed, by (3.28) we can choose a large enough to have:

$$\exp[-t \inf_{|y| \geqslant a-2d} V(y)] < \epsilon/2. \tag{3.31}$$

Now, if $|x| \leqslant a - d$ we have:

$$E_{W_x} \left\{ \exp\left[-\int_0^t V(X_s) \, ds \right]; \, |X_t| > a \right\} < \epsilon,$$

because of (3.30), and, if $|x| > a - d$ we have:

$$E_{W_x}\left\{\exp\left[-\int_0^t V(X_s)\,ds\right]; |X_t| > a\right\}$$

$$\leqslant E_{W_x}\left\{\exp\left[-\int_0^t V(X_s)\,ds\right]; \sup_{0\leqslant s\leqslant t} |x - X_s| > d\right\}$$

$$+ E_{W_x}\left\{\exp\left[-\int_0^t V(X_s)\,ds\right]; \sup_{0\leqslant s\leqslant t} |x - X_s| \leqslant d\right\}$$

$$< \epsilon,$$

by the conjunction of (3.30) and (3.31). ∎

PROPOSITION 3.4. *Let* $V = V_1 - V_2$ *be a potential function of class* $\mathscr{V}$ *such that*:

$$\lim_{|x|\to\infty} V_1(x) = +\infty.$$

Then, for each $t > 0$ *and each* $q \in [1, \infty]$, T_t *is a compact operator on* L^q.

Proof. Since an operator is compact if and only if its adjoint is compact, we can assume $1 \leqslant q \leqslant 2$. By Proposition 3.1 we have, for each $a > 0$:

$$\sup_{\|f\|_q\leqslant 1} \sup_{|x|\leqslant a} |[T_t f](x)| < +\infty. \tag{3.32}$$

Now, let α be any exponent satisfying $1 < \alpha < \infty$ and let α' be its conjugate exponent. For each $a > 0$ we have:

$$\int_{|x|>a} |[T_t f](x)|^q\,dx$$

$$\leqslant \int_{|x|>a} E_{W_x}\left\{|f(X_t)|^q \exp\left[-q\int_0^t V(X_s)\,ds\right]\right\} dx$$

$$= \int_{\mathbb{R}^n} |f(x)|^q E_{W_x}\left\{\exp\left[-q\int_0^t V(X_s)\,ds\right]; |X_t| > a\right\} dx$$

$$\leqslant \|f\|_q^q \sup_{x\in\mathbb{R}^n} E_{W_x}\left\{\exp\left[-q\alpha\int_0^t V_1(X_s)\,ds\right]; |X_t| > a\right\}^{1/\alpha}$$

$$\times \sup_{x\in\mathbb{R}^n} E_{W_x}\left\{\exp\left[\alpha'q\int_0^t V_2(X_s)\,ds\right]\right\}^{1/\alpha'},$$

which, by Theorem 2.1 and Lemma 3.3, implies:

$$\lim_{a\to\infty} \sup_{\|f\|_q\leqslant 1} \int_{|x|>a} |[T_t f](x)|^q\,dx = 0. \tag{3.33}$$

(3.32) and (3.33) are sufficient for T_t to be compact on L^q if $q \neq 1$. Indeed a proof works as follows: if we set:

$$^aT_t^1 f = (T_t f)\, \mathbb{1}_{\{|x| \leqslant a\}} \qquad \text{and} \qquad ^aT_t^2 f = T_t f - {}^aT_t^1 f,$$

$^aT_t^1$, when considered as an operator from L^q into $L^q(\{|x| \leqslant a\}, dx)$, factorizes through $L^\infty(\{|x| \leqslant a\}, dx)$ because of (3.32); since the natural embedding of $L^\infty(\{|x| \leqslant a\}, dx)$ into $L^q(\{|x| \leqslant a\}, dx)$ is completely continuous (i.e. maps weakly convergent sequences into norm convergent ones), and since L^q is reflexive, $^aT_t^1$ is compact and maps the closed unit ball of L^q into a totally bounded set. Now, (3.33) provides us with a control on the image of the closed unit ball of L^q by $^aT_t^2$, and this control makes possible checking that T_t maps the closed unit ball of L^q into a totally bounded subset of L^q.

This argument does not work when $q = 1$ because L^q is no longer reflexive. Nevertheless, the following trick can be used: by Proposition 3.1 T_t is bounded from L^1 into L^r for any $r > 1$ and any $t > 0$. Now, using the formulae

$$^aT_t^1 f = {}^aT_{t/2}^1 [T_{t/2} f]$$

we can conclude as above. ∎

Remark 3.7.   Let us note that the assumptions on V are weaker than:

$$V \in \mathscr{V} \qquad \text{and} \qquad \lim_{|x| \to \infty} V_+(x) = \infty.$$

IV. Schrödinger and Dirichlet Operators

IV.1. *Schrödinger Operators as Infinitesimal Generators*

One of the first tasks of nonrelativistic quantum mechanics is to construct self-adjoint extensions of the formal differential operator $-\tfrac{1}{2}\Delta + V$ for fairly general potential functions V. Let us show how the infinitesimal generators $-H_q$ constructed in the preceding section, supply us with such reasonable extensions.

PROPOSITION 4.1.   *Let $q \in [1, \infty[$, let q' denote its conjugate exponent and let V be a potential function of class $\mathscr{V}$.*

(i)   *If $V \in L_{loc}^{q'}$ we have:*

$$\mathscr{D}(H_q) \subset \{f \in L^q;\ -\tfrac{1}{2}\Delta f + Vf \in L^q\} \quad \text{and} \quad f \in \mathscr{D}(H_q) \Rightarrow H_q f = -\tfrac{1}{2}\Delta f + Vf.$$
$$(4.1)$$

Moreover, if V is bounded below, inclusion sign can be replaced by equality sign.

(ii) *If* $V \in L_{loc}^q$ *we have*:

$$C_c^\infty(\mathbb{R}^n) \subset \mathscr{D}(H_q) \qquad and \qquad f \in C_c^\infty(\mathbb{R}^n) \Rightarrow H_q f = -\tfrac{1}{2} \Delta f + Vf.$$

Proof. Let us assume first that V is bounded below and, in order to prove (i) let us fix $f \in \mathscr{D}(H_q)$. For each integer $k \geqslant 1$ let us define the truncated potential function V_k by:

$$V_k = V \mathbb{1}_{\{V \leqslant k\}} . \tag{4.2}$$

V_k is bounded and $T_t^{V_k}$ satisfies (3.16) with E independent of k (for example we can choose $E = \max\{0, -\inf V\}$). Let us denote by $-H_{q,k}$ the infinitesimal generator of the semigroup $\{T_t^{V_k}; t \geqslant 0\}$ acting on L^q, and let $\lambda > E$ be fixed. If $g \in L^q$ is defined by:

$$f = (\lambda + H_q)^{-1} g \tag{4.3}$$

we set:

$$f_k = (\lambda + H_{q,k})^{-1} g \qquad k = 1, 2,\dots . \tag{4.4}$$

For each $k \geqslant 1$, $f_k \in \mathscr{D}(H_{q,k})$, and since V_k is bounded, $\mathscr{D}(H_{q,k}) = \mathscr{D}(A_q)$, which implies:

$$\tfrac{1}{2} \Delta f_k = V_k f_k + \lambda f_k - g. \tag{4.5}$$

If we let k go to infinity in (4.5), the right hand side converges in L_{loc}^1 to the function $Vf + \lambda f - g$, and the distribution of the left hand side converges weakly to the distribution $\tfrac{1}{2} \Delta f$, which, together with (4.3) prove (4.1). Let us note that we used essentially the fact that f_k converges in L^q to f, which is due to the approximation argument we mentioned in section III.

In order to prove equality instead of inclusion in (4.1) let us define the operator T_q on L^q by:

$$\mathscr{D}(T_q) = \{f \in L^q; -\tfrac{1}{2} \Delta f + Vf \in L^q\} \quad and \quad f \in \mathscr{D}(T_q) \Rightarrow T_q f = -\tfrac{1}{2} \Delta f + Vf.$$

We already proved $H_q \subset T_q$. Now, let us fix $f \in \mathscr{D}(T_q)$ and let us define $g \in L^q$ and $h \in L^q$ by:

$$g = (\lambda + T_q)f \qquad and \qquad g = (\lambda + H_q)h,$$

where λ is the same as above. Setting $\varphi = f - h$ we obtain:

$$(T_q + \lambda)\varphi = 0.$$

Since $\varphi \in L^q$ and since its Laplacian in the sense of distributions is in L_{loc}^1, we can use Kato's inequality (see for example [32, Theorem X.27]) and obtain:

$$\Delta |\varphi| \geqslant (V + \lambda) |\varphi| \geqslant 0.$$

A standard argument (see for example the end of the proof of [32, Theorem X.28]) assures that $\varphi = 0$ which implies $f = h \in \mathscr{D}(H_q)$. This concludes the proof of (i) when V is bounded below. In order to prove (i) for unbounded below potentials V, we use as before, a classical approximation argument. For each integer $k \geqslant 1$ we define the truncated potential function V_k by:

$$V_k = V \mathbb{1}_{\{V \geqslant -k\}} . \tag{4.6}$$

Now we choose a positive constant E which is independent of k and which satisfies (3.16) for all the semigroups $\{T_t^{V_k}; \; t \geqslant 0\}$. If, as before, we let $-H_{q,k}$ denote the infinitesimal generator of the semigroup $\{T_t^{V_k}; \; t \geqslant 0\}$ acting on L^q, and λ be fixed such that $\lambda > E$, again $-\lambda$ is in the resolvent set of H_q and all the $H_{q,k}$, and our approximation argument of section III implies strong convergence of the corresponding resolvent operators. We end the proof as above. Namely we fix f in $\mathscr{D}(H_q)$, we define g and the f_k by (4.3) and (4.4) and we let k go to infinity in (4.5) which remains true because V_k is bounded below and because we already proved (i) for such potential functions.

Now, let us assume $V \in L_{\mathrm{loc}}^q$ and let us prove (ii). As before we assume first that V is bounded below. For λ as before, $\psi = \lambda\varphi - \frac{1}{2}\varDelta\varphi + V\varphi$ is in L^q and:

$$\lim_{k \to \infty}(\lambda + H_{q,k})^{-1}\psi = (\lambda + H_q)^{-1}\psi \tag{4.7}$$

in L^q sense, where $H_{q,k}$ is the truncated operator given by the potential V_k of (4.2). Moreover we have $\varphi \in \mathscr{D}(H_{q,k})$ and if we set:

$$\psi_k = (\lambda + H_{q,k})\varphi \qquad k \geqslant 1,$$

then we have:

$$\lim_{k \to \infty}(\lambda + H_{q,k})^{-1}\psi = \varphi, \tag{4.8}$$

because $V \in L_{\mathrm{loc}}^q$ and because by (3.16) and the choice of λ the resolvent operator norms $\|(\lambda + H_{q,k})^{-1}\|$ are uniformly bounded in k. The conjunction of (4.7) and (4.8) prove $\psi = \lambda\varphi + H_q\varphi$ which concludes the proof when V is bounded below. Now, the general case follows from the same proof provided V_k is defined by (4.6) instead of (4.2). ∎

Remark 4.1.   Results of Proposition 4.1 are standard in the case $q = 2$ and V bounded below (see for example [2, 14, 27]). Schrödinger operator in L^q was also defined and studied in [37] where results similar to those of Proposition 4.1 are proved in much the same way.

Remark 4.2.   By (3.16), H_q is bounded below and (3.2) provides us with a lower bound on the bottom, say E_0, of its spectrum. Namely we have:

$$E_0 \geqslant \inf V_1 - c(p)^{1/\epsilon}\| V_2 \|_p^{1/\epsilon}\varGamma(\epsilon)^{1/\epsilon}$$

IV.2. *Schrödinger Operator as Form Sum*

A by now standard way to define perturbed Hamiltonians in mathematical physics, and especially in quantum mechanics, is to consider them as self-adjoint operators associated to the sum of quadratic forms (see, for example, [25, Chapter VI], [40, 10] and [32, Chapter X]). We briefly review this technique of definition, and we check that Schrödinger operator defined in this way coincides with the self-adjoint infinitesimal generator H_2 which we introduced in section III.

If f and g belong to $C_c^2(\mathbb{R}^n)$ let us set:

$$\epsilon_0(f, g) = \frac{1}{2} \int_{\mathbb{R}^n} \nabla f(x) \cdot \overline{\nabla g(x)} \, dx \tag{4.9}$$

Integrating by parts the right hand side of (4.9) we obtain:

$$\epsilon_0(f, g) = (-\tfrac{1}{2} \Delta f, g) \tag{4.10}$$

where (,) denotes the L^2-inner product. This proves that the form ϵ_0 is closable (we will use the same symbol ϵ_0 to denote the closure). The form domain $Q(\epsilon_0)$ of ϵ_0 is the set of elements of L^2 the first order derivatives of which (in the sense of distributions) are elements of L^2.

Now, let V be a real valued measurable function on $\mathbb{R}^n$, and let us set:

$$Q(V) = \left\{ f \in L^2; \int_{\mathbb{R}^n} |V(x)| \, |f(x)|^2 \, dx < +\infty \right\}.$$

On $Q(\epsilon_0) \cap Q(V)$ we define the form ϵ by:

$$\epsilon(f, g) = \epsilon_0(f, g) + (Vf, g)$$

where (Vf, g) denotes $((\text{sgn } V) \, | \, V \, |^{1/2}f, \, | \, V \, |^{1/2}g)$. If we assume further that V possesses a breakup $V = V_1 - V_2$ such that:

$$V_1 \text{ is bounded below and } V_1 \in L^1_{\text{loc}}$$
$$\tag{4.11}$$
$$V_2 \in L^p \text{ for some } p \geqslant n/2 \text{ if } n \geqslant 3, \, p > 1 \text{ if } n = 2, \text{ and } p = 1 \text{ if } n = 1$$

Then, by [10, p. 27], ϵ is the form of a unique bounded below self-adjoint operator H on L^2, the form domain of which is given by $Q(H) = Q(\epsilon_0) \cap Q(V)$. It is known (see for example [12]) that monotone convergence theorems for integrals and for quadratic forms can be used to extend the classical Feynman–Kac formula (which is easy to prove for bounded V) to general potential functions V which satisfy the above assumptions; namely, for each $t > 0$, for each $f \in L^2$ and for almost every $x \in \mathbb{R}^n$ we have:

$$[e^{-tH}f](x) = E_{W_x} \left\{ f(X_t) \exp \left[-\int_0^t V(X_s) \, ds \right] \right\}.$$

Consequently, if $V \in \mathscr{V}$ the semigroup generated by H and the semigroup $\{T_t \, ; \, t \geqslant 0\}$ studied in Section III coincide and thus their infinitesimal generators are identical. The following remarks are designed to compare the results we proved to those existing in the literature.

Remark 4.3. When $q = 2$ and $V \in \mathscr{V}$ is bounded below, the result of Proposition 4.1 is weaker than [32, Theorem X.32]; indeed in this case, the very domain of the operator H defined as sum of quadratic forms can be explicitly given when $V \in L^1_{\text{loc}}$ rather than when $V \in L^2_{\text{loc}}$ as we had to assume.

Remark 4.4. We will assume later on that the infimum of the spectrum of H is an eigenvalue. In this case it is very important to know wether or not $\exp[-tH]$ is positivity improving (i.e. $\exp[-tH]f > 0$ a.e. whenever $f \geqslant 0$ a.e. and $f \neq 0$). Indeed, when this is true we can conclude that the infimum of the spectrum of H is a simple eigenvalue and that the corresponding normalized eigenfunction is positive a.e.. We would like to point out that once Feynman–Kac's formula is known, the positivity improving property comes out naturally from the following simple argument: if $f \in L^2$ is non-negative a.e. it is clear that $\exp[-tH]f$ is non-negative a.e.. Moreover, for each Borel subset A of $\mathbb{R}^n$,

$$\cdot \qquad \forall x \in A, \qquad E_{W_x} \left\{ f(X_t) \exp\left[-\int_0^t V(X_s)\, ds \right] \right\} = 0.$$

implies that for all x such that $\int_0^t V(X_s)\, ds < +\infty$ W_x a.s. we have:

$$E_{W_x}\{f(X_t)\} = 0,$$

which implies that A has Lebesgue's measure zero. This proves that $\exp[-tH]$ is positivity improving. Furthermore, if V_1 is assumed to be in $L^{+n/2}_{\text{loc}}$, the above argument shows that $\exp[-tH]f$ may be chosen everywhere positive whenever $f \geqslant 0$ and f is not a.e. equal to zero. Consequently, the groundstate (i.e. the eigenfunction associated to the infimum of the spectrum of H) is positive and locally bounded away from zero since, by Proposition 3.3 it is known to be continuous. We emphasize this simple property because it was used in several places (see for example [9]) and proved to hold under restrictive conditions by rather lengthy analytical computations.

Remark 4.5. As before we assume that V possesses a breakup $V = V_1 - V_2$ satisfying (4.11) and such that $V_1 \in L^2_{\text{loc}}$ if $n \leqslant 3$ and $V_1 \in L^{+n/2}_{\text{loc}}$ if $n \geqslant 4$. Since for all n we have $V_1 \in L^2_{\text{loc}}$, we know that $-\frac{1}{2}\Delta + V_1$ defined as a sum of operators on $C_c^\infty(\mathbb{R}^n)$ is essentially self-adjoint (see [32, Theorem X.28]). By [32, Theorem X.28], its unique self-adjoint extension is nothing but $-\frac{1}{2}\Delta + V_1$ defined as sum of quadratic forms. Now, classical Sobolev's

inequalities imply that V_2 is a small form perturbation of $-\frac{1}{2}\Delta$, and thus of $-\frac{1}{2}\Delta + V_1$ too (see [10, Theorem 9.2]). Consequently, by [32, Theorem X.17], not only we recover the possibility of defining $H = -\frac{1}{2}\Delta + V_1 - V_2$ as sum of quadratic forms, but we learn that $C_c^\infty(\mathbb{R}^n)$ is a form core for the so-obtained bounded below self-adjoint operator. We will need to know another form core for H. Namely:

PROPOSITION 4.2. *Let V be as above and let us assume that ψ is any nonnegative function in $Q(\epsilon_0)$ which is bounded and locally bounded away from zero. Then $\{\psi f; f \in C_c^\infty(\mathbb{R}^n)\}$ is a form core for H.*

Proof. It suffices to prove that for each $f \in C_c^\infty(\mathbb{R}^n)$ there is a sequence $\{f_k \, ; \, k \geqslant 1\}$ in $C_c^\infty(\mathbb{R}^n)$ such that ψf_k converges to f in the H-graph norm. The negative part of V possesses a breakup $V_- = U + W$ with U bounded and W a small form perturbation of $-\frac{1}{2}\Delta + V_+$. Consequently we may assume that V is non-negative. Let us remark that $f\psi^{-1} \in Q(H) = Q(\epsilon_0) \cap Q(V)$ whenever $f \in C_c^\infty(\mathbb{R}^n)$. Indeed, we have first:

$$\int_{\mathbb{R}^n} V(x) |f(x) \psi^{-1}(x)|^2 \, dx \leqslant \|f\psi^{-1}\|_\infty^2 \int_{\{f(x) \neq 0\}} V(x) \, dx < +\infty,$$

because ψ is locally bounded away from zero and $V \in L_{\text{loc}}^1$, and second, for each $i \in \{1,...,n\}$:

$$\int_{\mathbb{R}^n} \left| \frac{\partial(f\psi^{-1})}{\partial x_i}(x) \right|^2 dx$$

$$\leqslant 2 \sup_{f(x) \neq 0} \psi(x)^{-2} \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_i}(x) \right|^2 dx + 2 \|f\psi^{-2}\|_\infty^2 \int_{\mathbb{R}^n} \left| \frac{\partial \psi}{\partial x_i}(x) \right|^2 dx$$

$$< +\infty,$$

because ψ is locally bounded away from zero and because $\psi \in Q(\epsilon_0)$. Thereby we can find a sequence $\{f_k \, ; \, k \geqslant 1\}$ in $C_c^\infty(\mathbb{R}^n)$ which converges to $f\psi^{-1}$ in H-graph norm. Since $V \in L_{\text{loc}}^{n/2}$ and since the support of $f\psi^{-1}$ is compact, this sequence can be obtained by convolution with an approximate identity (see the proof of Proposition 4.3 below for a definition), and consequently we may assume without any loss of generality that (a) the f_k are uniformly bounded, (b) their supports are contained in a single compact subset K of $\mathbb{R}^n$, (c) they converge almost everywhere to $f\psi^{-1}$. Now, on one hand we have:

$$\int_{\mathbb{R}^n} |f(x) - \psi(x) f_k(x)|^2 \, dx \leqslant \|\psi\|_\infty^2 \int_{\mathbb{R}^n} |f(x) \psi^{-1}(x) - f_k(x)|^2 \, dx$$

which goes to zero when k goes to infinity, and on the other hand:

$$(H[f - \psi f_k], f - \psi f_k)$$

$$= \int_{\mathbb{R}^n} |\nabla(f - \psi f_k)(x)|^2 \, dx + \int_{\mathbb{R}^n} V(x) |f(x) - \psi(x) f_k(x)|^2 \, dx$$

$$= \int_{\mathbb{R}^n} |\nabla(f\psi^{-1} - f_k)(x)|^2 \, \psi(x)^2 \, dx + \int_{\mathbb{R}^n} V(x) |(f\psi^{-1} - f_k)(x)|^2 \, \psi^2(x) \, dx$$

$$+ \int_{\mathbb{R}^n} |\nabla\psi(x)|^2 |(f\psi^{-1} - f_k)(x)|^2 \, dx,$$

which goes to zero when k goes to infinity because first, the sum of the first two terms of the above right hand side goes to zero by the boundedness of ψ and the construction of the sequence $\{f_k \, ; \, k \geqslant 1\}$ and second, the third term goes to zero by Lebesgue's dominated convergence theorem. This concludes the proof. ∎

IV.3. *Dirichlet Operators*

The theory of Dirichlet forms and Dirichlet spaces has been initiated by Beurling and Deny (see [4, 7]), and has been extensively studied since (see for example [15, 16, 38, 39, 1], and the references therein). In this subsection we will concentrate on Dirichlet forms that are unitary equivalent to the quadratic forms associated to the self-adjoint operators we introduced in subsection IV.1.

Let μ be a Borel probability measure on $\mathbb{R}^n$ which satisfies:

$$d\mu(x) = e^{-2h(x)} \, dx \qquad (4.12)$$

for some real-valued, locally bounded, absolutely continuous function h, the first order partial derivatives of which are in L^2_{loc}. If for all f and g in $C_c^\infty(\mathbb{R}^n)$ we set:

$$\delta(f, g) = \frac{1}{2} \int_{\mathbb{R}^n} \nabla f(x) \cdot \overline{\nabla g(x)} \, d\mu(x), \qquad (4.13)$$

integrating by parts the right hand side we obtain:

$$\delta(f, g) = (Df, g)_\mu \qquad (4.14)$$

where $(\ ,\ )_\mu$ denotes the $L^2(\mu)$-inner product and where:

$$Df = -\tfrac{1}{2}\Delta f + \nabla h \cdot \nabla f.$$

By (4.14) δ is given by a symmetric operator on $L^2(\mu)$ and thus δ is closable. We will use the same symbol δ to denote the closure of δ, and we will call it the Dirichlet form of μ.

PROPOSITION 4.3. *The form domain $Q(\delta)$ is the set of elements of $L^2(\mu)$ whose first order derivatives (in the sense of distributions) are in $L^2(\mu)$.*

Proof. Let $f \in Q(\delta)$ be fixed. There is a sequence $\{f_k \, ; \, k \geqslant 1\}$ in $C_c^\infty(\mathbb{R}^n)$ which satisfies:

$$\lim_{k \to \infty} f_k = f$$

in $L^2(\mu)$ and:

$$\lim_{h,k \to \infty} \delta(f_h - f_k, f_h - f_k) = 0. \tag{4.15}$$

(4.15) implies that for each $i \in \{1,\dots,n\}$, $\{\partial f_k / \partial x_i \, ; \, k \geqslant 1\}$ is a Cauchy sequence in $L^2(\mu)$. Since h is locally bounded, e^{-2h} is locally bounded away from zero and convergence in $L^2(\mu)$ implies convergence in L^2_{loc}, and thus in the sense of distributions. Consequently, $\partial f/\partial x_i$ in the sense of distributions is equal, almost everywhere, to the limit in $L^2(\mu)$ of the sequence $\{\partial f_k / \partial x_i \, ; \, k \geqslant 1\}$. Conversely, in order to prove that each function in $L^2(\mu)$ whose first order derivatives (in the sense of distributions) are in $L^2(\mu)$, is necessarily in $Q(\delta)$ we need to introduce some notations.

Let $\{j_\alpha \, ; \, \alpha > 0\}$ be an approximate identity; this means that we have:

$$j_\alpha(x) = \alpha^{-n} j(\alpha^{-1} x) \qquad \alpha > 0, \quad x \in \mathbb{R}^n,$$

for some non-negative C^∞-function on $\mathbb{R}^n$ j, the support of which is contained in $[-1, +1]^n$, and which satisfies:

$$\int_{\mathbb{R}^n} j(x) \, dx = 1.$$

Let $\{\theta_k \, ; \, k \geqslant 1\}$ be a sequence of C^∞-functions on $\mathbb{R}^n$ which satisfies: $0 \leqslant \theta_k \leqslant 1$, $\theta_k(x) = 1$ if $|x| \leqslant k$, $\theta_k(x) = 0$ if $|x| \geqslant k + 1$ and $\partial\theta/\partial x_i(x)$ is uniformly bounded for $x \in \mathbb{R}^n$, $k \geqslant 1$ and $i \in \{1,\dots,n\}$, and for each function φ on $\mathbb{R}^n$ let us set $\varphi^{(k)} = \varphi \theta_k$ for all integers $k \geqslant 1$.

Let $f \in L^2(\mu)$. For each integer $k \geqslant 1$ and each real $0 < \alpha < 1$, $j_\alpha * f^{(k)} \in C_c^\infty(\mathbb{R}^n)$ and its support is contained in $\{x \in \mathbb{R}^n; \, |x| \leqslant k + 2\}$. Moreover:

$$\int_{\mathbb{R}^n} |f - j_\alpha * f^{(k)}|^2 \, d\mu$$

$$\leqslant 2 \int_{\mathbb{R}^n} |f(1 - \theta_k)|^2 \, d\mu + 2 \int_{\mathbb{R}^n} |f^{(k)} - j_\alpha * f^{(k)}|^2 \, d\mu$$

$$\leqslant 2 \int_{\{|x| > k\}} |f|^2 \, d\mu + 2 \sup_{|x| \leqslant k+2} e^{-2h(x)} \int_{\mathbb{R}^n} |f^{(k)} - j_\alpha * f^{(k)}|^2 \, dx.$$

Thereby, if $\epsilon > 0$ is given, we can choose k large enough to have the first term less than $\epsilon/2$ and, once such a k is fixed, we can choose α small enough to have the second term less than $\epsilon/2$ too. Consequently we can choose a sequence of integers $\{k(h); h \geqslant 1\}$ and a sequence of real numbers $\{\alpha(h); h \geqslant 1\}$ which satisfy:

$$\lim_{h \to \infty} k(h) = \infty \qquad \text{and} \qquad \lim_{h \to \infty} \alpha(h) = 0,$$

and:

$$\lim_{h \to \infty} \| f - j_{\alpha(h)} * f^{(k(h))} \|_{L^2(\mu)} = 0. \tag{4.16}$$

Let us assume further that the first order derivatives of f (in the sense of distributions) are in $L^2(\mu)$. Since:

$$\frac{\partial}{\partial x_i} j_\alpha * f^{(k)} = j_\alpha * \left[\theta_k \frac{\partial f}{\partial x_i} \right] + j_\alpha * \left[f \frac{\partial \theta_k}{\partial x_i} \right],$$

the numbers $k(h)$ and $\alpha(h)$ could have been chosen in order to have, in addition to (4.16):

$$\lim_{h \to \infty} \left\| \frac{\partial f}{\partial x_i} - \frac{\partial}{\partial x_i} \left[j_{\alpha(h)} * f^{(k(h))} \right] \right\|_{L^2(\mu)} = 0 \qquad i = 1,\ldots,n. \tag{4.17}$$

Indeed we have:

$$\int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_i} - \frac{\partial}{\partial x_i} j_\alpha * f^{(k)} \right|^2 d\mu$$

$$\leqslant 4 \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_i} (1 - \theta_k) \right|^2 d\mu + 4 \int_{\mathbb{R}^n} \left| f \frac{\partial \theta_k}{\partial x_i} \right|^2 d\mu$$

$$+ 4 \int_{\mathbb{R}^n} \left| \theta_k \frac{\partial f}{\partial x_i} - j_\alpha * \left[\theta_k \frac{\partial f}{\partial x_i} \right] \right|^2 d\mu$$

$$+ 4 \int_{\mathbb{R}^n} \left| j_\alpha * \left[f \frac{\partial \theta_k}{\partial x_i} \right] - f \frac{\partial \theta_k}{\partial x_i} \right|^2 d\mu,$$

and if $\epsilon > 0$ is given, we can choose k large enough to have the sum of the first two terms less than $\epsilon/2$. Now, once such a k is fixed, we can choose α small enough to have the sum of the last two terms less than $\epsilon/2$ too because e^{-2h} is locally bounded, $\theta_k(\partial f/\partial x_i)$ and $f(\partial \theta_k/\partial x_i)$ are in L^2 and consequently $j_\alpha * [\theta_k(\partial f/\partial x_i)]$ and $j_\alpha * [f(\partial \theta_k/\partial x_i)]$ converge in L^2 to $\theta_k(\partial f/\partial x_i)$ and $f(\partial \theta_k/\partial x_i)$ respectively when α goes to zero. (4.17) is proved, and the conjunction of (4.16) and (4.17) proves $f \in Q(\delta)$ and

$$\delta(f, f) = \int_{\mathbb{R}^n} | \nabla f(x)|^2 \, d\mu(x). \quad \blacksquare$$

EXAMPLE 4.1. Let $V = V_1 - V_2$ with V_1 bounded below, $V_1 \in L_{\mathrm{loc}}^{+n/2}$, $V_2 \geqslant 0$ and $V_2 \in L^p$ for some $p > \max\{1, n/2\}$. Let us assume that the infimum of the spectrum of H, say E, is an eigenvalue, and let us denote by ψ the corresponding groundstate eigenfunction. If we define the Borel probability measure μ on $\mathbb{R}^n$ by:

$$d\mu(x) = e^{-2h(x)}\, dx, \qquad (4.18)$$

where $h = -\mathrm{Log}\,\psi$, and if we define the operator $\bar{D}$ by:

$$\bar{D} = C(H - E)\, C^{-1},$$

where C is the unitary from L^2 onto $L^2(\mu)$ defined by:

$$C\varphi = \psi^{-1}\varphi, \qquad \varphi \in L^2,$$

then, $\bar{D}$ is a positive self-adjoint operator in $L^2(\mu)$, 0 is a simple eigenvalue and the constant function $\mathbb{1}$ is the corresponding eigenfunction. In fact $\bar{D}$ is the unique positive self-adjoint operator associated to the closed positive bilinear form, say $\bar{\delta}$, corresponding to $\epsilon - E$ in the unitary equivalence C, and consequently, by Proposition 4.2 $C_c^\infty(\mathbb{R}^n)$ is a core for $\bar{\delta}$.

Now, since ψ is bounded and locally bounded away from zero, h is bounded below and locally bounded above. Moreover, since $\psi \in Q(\epsilon_0)$, the first order partial derivatives of h are in L_{loc}^2, and consequently we can associate to μ defined by (4.18) a Dirichlet form, say δ, and a Dirichlet operator, say D. Since δ is defined as the closure of a form the domain of which is $C_c^\infty(\mathbb{R}^n)$ and since $C_c^\infty(\mathbb{R}^n)$ is a form core for $\bar{\delta}$, in order to prove $\delta = \bar{\delta}$ (and thus $D = \bar{D}$) it suffices to prove that δ and $\bar{\delta}$ coincide on $C_c^\infty(\mathbb{R}^n)$. But, if $f \in C_c^\infty(\mathbb{R}^n)$ we have:

$$\bar{\delta}(f,f) = [\epsilon - E](\psi f, \psi f)$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} |\nabla(\psi f)|^2\, dx + \int_{\mathbb{R}^n} V\,|f|^2\psi^2\, dx - E\int_{\mathbb{R}^n} |f|^2\psi^2\, dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \nabla\psi \cdot \nabla(\psi\,|f|^2)\, dx + \int_{\mathbb{R}^n} V\psi\psi\,|f|^2\, dx - E\int_{\mathbb{R}^n} \psi\psi\,|f|^2\, dx$$

$$\quad - \frac{1}{2}\int |\nabla f|^2\psi^2\, dx$$

$$= ([\epsilon - E]\,\psi, \psi\,|f|^2) + \delta(f,f)$$

$$= \delta(f,f).$$

Thus the Dirichlet form and the Dirichlet operator of the groundstate measure are unitary equivalent to the Schrödinger form and the Schrödinger operator respectively. Similar results were obtained in [1] under conditions that seem to be more restrictive than ours.

V. Hyper and Super Contractivity of Dirichlet Semigroups

Let $V \in L_{loc}^{+n/2}$ be a potential of class $\mathscr{V}$ and as before let $H = H_2$ be either the infinitesimal generator of the semigroup $\{T_t ; t \geqslant 0\}$ acting in L^2, or the operator $-\frac{1}{2}\Delta + V$ defined as sum of quadratic forms. As before *we assume that the infimum of the spectrum of H, say E, is an eigenvalue*. We denote by ψ the groundstate measure (i.e. $d\mu(x) = \psi(x)^2 dx$), by D the Dirichlet operator of μ and we set $h = -\mathrm{Log}\,\psi$. In this section we investigate the hyper and supercontractive properties of the Dirichlet semigroup $\{e^{-tD}; t \geqslant 0\}$ on $L^2(\mu)$. These properties are defined as follows:

DEFINITION 5.1 [21, 33]. The semigroup $\{e^{-tD}; t \geqslant 0\}$ is said hypercontractive if for some $t > 0$ and some $r > 2$, e^{-tD} is a bounded operator from $L^2(\mu)$ into $L^r(\mu)$. It is said supercontractive if for all $t > 0$, $r > 1$ and $s > 1$, e^{-tD} is a bounded operator from $L^r(\mu)$ into $L^s(\mu)$.

In order to tackle the hyper (or super) contractivity problem, we use the logarithmic Sobolev inequalities approach which was discovered and proved to be relevant by Gross [19]. First let us recall a definition of his:

DEFINITION 5.2. The operator D is called a Sobolev generator if for some real constants $c > 0$ and γ there holds:

$$\int_{\mathbb{R}^n} |f|^2 \, \mathrm{Log}\,|f| \, d\mu \leqslant c(Df,f)_\mu + \gamma \|f\|_{L^2(\mu)}^2 + \|f\|_{L^2(\mu)}^2 \, \mathrm{Log}\,\|f\|_{L^2(\mu)} \qquad (5.1)$$

for each $f \in \mathscr{D}(D)$. The constants c and γ are called the Sobolev coefficient and the local norm of D.

Remark 5.1. From Fatou's lemma, relation (5.1) can be equivalently required for f in the form domain $Q(D)$ instead of the operator domain $\mathscr{D}(D)$, or simply in any form core of D.

The relevance of the concept of Sobolev generator to our problem is contained in the following:

PROPOSITION 5.1. $\{e^{-tD}; t \geqslant 0\}$ *is hyper (resp. super) contractive if and only if D is a Sobolev generator (resp. with Sobolev coefficient arbitrarily small).*

Proof. The "only if" part is contained in [19, Example 2]. The if part is a consequence of the Eckman–Rosen version of Gross' fundamental result [19, Theorem 6] provided we check that the assumption "the set of bounded twice continuously differentiable functions with bounded first and second derivatives is a core for D" is unnecessary in the present situation. Now, using Remark 5.1 and by now standard properties of derivatives in the sense of

distributions (see for example [35], or [8, Theorem 3.2]), we can easily push Eckman's version ([9, Lemma 3.3]) of [19, Lemma 6] to obtain:

If μ is a probability measure on $\mathbb{R}^n$ which satisfies:

$$\int_{\mathbb{R}^n} |f|^2 \operatorname{Log} |f| \, d\mu \leqslant c \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu + \gamma \|f\|_{L^2(\mu)}^2 + \|f\|_{L^2(\mu)}^2 \operatorname{Log} \|f\|_{L^2(\mu)}$$

for some constants $c > 0$ and $\gamma \geqslant 0$, and all f in the set $\mathscr{D}_1$ of bounded functions with first order derivatives (in the sense of distributions) in $L^2(\mu)$, then for all r in $[2, +\infty[$ we have:

$$\int_{\mathbb{R}^n} |f|^r \operatorname{Log} |f| \, d\mu \leqslant c(r) \int_{\mathbb{R}^n} \nabla f \cdot \nabla f_r \, d\mu + \gamma \|f\|_{L^2(\mu)}^r + \|f\|_{L^2(\mu)}^r \operatorname{Log} \|f\|_{L^2(\mu)}$$

$$\text{(l.s.i.)}$$

for all f in $\mathscr{D}_1$, where $f_r = (\operatorname{sgn} f) |f|^{r-1}$ and $c(r) = cr/2(r - 1)$.

Furthermore the proof of [19, Theorem 1] applies to the present situation once one makes the following two observations: (a) all is needed in the proof of [19, Theorem 1] is pluging functions f of the form $e^{-tD}g$ with $g \in L^\infty(\mu)$ in (l.s.i.), (b) $e^{-tD}g$ belongs to $\mathscr{D}_1$ whenever g is in $L^\infty(\mu)$ because first, e^{-tD} is a contraction in $L^r(\mu)$ for all r in $[1, \infty]$ and $t \geqslant 0$, and second, because of the conjunction of Proposition 4.3 and the fact that $e^{-tD}g \in \mathscr{D}(D) \subset Q(D)$.

Consequently we have:

$$\| e^{-tD} \|_{r, 1+(r-1)e^{2t/c}} \leqslant e^{\gamma t} \tag{5.2}$$

for all $t \geqslant 0$ and $r \geqslant 2$. Thus, e^{-tD} is a bounded operator from $L^r(\mu)$ into $L^s(\mu)$ if $r \geqslant 2$ and $e^{2t/c} \geqslant (s - 1)/(r - 1)$. This already proves that the semigroup $\{e^{-tD}; t \geqslant 0\}$ is hypercontractive if D is a Sobolev generator. Moreover, if the Sobolev coefficient c can be chosen arbitrarily small, the supercontractive property is obtained via (5.2), the self-adjointness of D and the semigroup property. ∎

Remark 5.2. If D is a Sobolev generator the three following properties are equivalent:

    (i) increasing the Sobolev coefficient if necessary, the local norm of D can be taken to be zero.

    (ii) there is a gap at the bottom of the spectrum of D.

    (iii) e^{-tD} is a contraction from $L^2(\mu)$ into $L^4(\mu)$ for t large.

The equivalence of (i) and (iii) is due to Gross [19, Example 2 and Theorem 6]. (ii) ⇒ (iii) is due to Glimm [17] and (iii) ⇒ (ii) to Simon [42].

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LEMMA 5.1. *If for some nonnegative constants a and b we have*:

$$| x |^2 \leqslant aD + b \tag{5.3}$$

in the sense of quadratic forms on $L^2(\mu)$, then:

$$\alpha < (n \max\{1, a, b\})^{-1} \Rightarrow \int_{\mathbb{R}^n} e^{\alpha|x|^2} d\mu(x) < \infty. \tag{5.4}$$

Proof. Let $j \in \{1,..., n\}$ be fixed. Since $\mathbb{1} \in Q(D)$, by (5.3) we have:

$$\int_{\mathbb{R}^n} x_j^2 \, d\mu(x) \leqslant \int_{\mathbb{R}^n} | x |^2 \, d\mu(x) \leqslant b < +\infty. \tag{5.5}$$

For each integer $k \geqslant 1$ let us define the function f_k on the real line by:

$$f_k(x) = \begin{cases} k & \text{if } k < x \\ x & \text{if } -k \leqslant x \leqslant k \\ -k & \text{if } x < -k. \end{cases}$$

If $m \geqslant 1$ is any integer, by Proposition 4.3 the function $f_k(x_j)^m$ on $\mathbb{R}^n$ is in the form domain of D. Consequently, by (5.3) we have:

$$\int_{A_{jk}} x_j^{2(m+1)} \, d\mu(x) \leqslant am^2 \int_{A_{jk}} x_j^{2(m-1)} \, d\mu(x) + b \int_{\mathbb{R}^n} f_k(x_j)^{2m} \, d\mu(x) \tag{5.6}$$

where $A_{jk} = \{(x_1,..., x_n) \in \mathbb{R}^n; | x_j | \leqslant k\}$. Using (5.5), a simple inductive argument and Fatou's lemma, we can let k go to infinity in (5.6) and obtain:

$$\int_{\mathbb{R}^n} x_j^{2(m+1)} \, d\mu(x) \leqslant am^2 \int_{\mathbb{R}^n} x_j^{2(m-1)} \, d\mu(x) + b \int_{\mathbb{R}^n} x_j^{2m} \, d\mu(x) \tag{5.7}$$

where both sides are finite. Now let $c = \max\{1, a, b\}$. By a simple inductive argument from (5.7) we obtain:

$$\int_{\mathbb{R}^n} x_j^{2(m+1)} \, d\mu(x) \leqslant am^2(m - 1)! \, c^{m-1} + bm! \, c^m \leqslant (m + 1)! \, c^{m+1}, \tag{5.8}$$

and from (5.8) and Fatou's lemma we have:

$$\int_{\mathbb{R}^n} e^{\alpha n x_j^2} \, d\mu(x) \leqslant \sum_{m=0}^{\infty} \frac{(\alpha n)^m}{m!} \int_{\mathbb{R}^n} x_j^{2m} \, d\mu(x)$$

$$\leqslant \sum_{m=0}^{\infty} (\alpha n c)^m$$

$$< \infty,$$

because we assumed $\alpha n c < 1$. This concludes the proof of the lemma. ∎

Remark 5.3. It is possible to manufacture counter examples to prove that the converse of Lemma 5.1 is false.

For later reference we restate [19, Theorem 7]. This semi-boundedness result was proved by L. Gross in the case of a Sobolev generator with local norm zero, but his proof works in the general case and yields:

LEMMA 5.2. *If D is a Sobolev generator with Sobolev coefficient c and local norm γ and if U is a real measurable function such that $\int e^U \, d\mu < \infty$, then we have:*

$$U \leqslant cD + \gamma + \frac{1}{2} \operatorname{Log} \int e^U \, d\mu$$

as quadratic forms on $L^2(\mu)$.

Remark 5.4. From Lemma 5.1 and Lemma 5.2 it follows that, if D is a Sobolev generator, the following properties are equivalent:

  (iv) $\int_{\mathbb{R}^n} e^{\alpha |x|^2} \, d\mu(x) < \infty$ for some positive constant α

  (v) $|x|^2 \leqslant aD + b$ for some real constant $a > 0$ and b.

This equivalence, at least when D has a gap at the bottom of its spectrum, is essentially due to I. W. Herbst who may have had a different proof. We learned this fact from L. Gross (oral communication).

We next prove that properties (i) to (v) hold whenever D is a Sobolev generator. First we need some simple lemma.

LEMMA 5.3. *If D is a Sobolev generator, then D has compact resolvent.*

Proof. Since we have:

$$\sum_{j=1}^{\infty} \int_{j-1 \leqslant |x| \leqslant j} \psi^2(x) \, dx < +\infty,$$

it is possible to choose a sequence $\{\alpha_j ; j \geqslant 1\}$ of positive numbers which satisfies:

$$\lim_{j \to \infty} \alpha_j = +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} \alpha_j \int_{j-1 \leqslant |x| \leqslant j} \psi^2(x) \, dx < +\infty. \tag{5.9}$$

Now, let us define the function U by:

$$U = \sum_{j=1}^{\infty} (\operatorname{Log} \alpha_j) \, \mathbb{1}_{\{j-1 \leqslant |x| \leqslant j\}} \, .$$

By (5.9) we have $\int e^U \, d\mu < \infty$, which implies, by Lemma 5.2 that:

$$U \leqslant a_1 D + b_1 \tag{5.10}$$

for some real constants $a_1 > 0$ and b_1. Since D is unitary equivalent to $H = -\frac{1}{2}\Delta + V$ via a multiplication operator, (5.10) can be restated in:

$$U \leqslant a_1 H + b_1$$

as quadratic forms on L^2, and this implies that:

$$-\tfrac{1}{2}\Delta + U \leqslant aH + b \qquad\qquad (5.11)$$

for some constants $a > 0$ and b. Finally, together with $\lim_{|x|\to\infty} U(x) = +\infty$, (5.11) implies the desired conclusion. ∎

Following some suggestion of I. W. Herbst, L. Gross gave a simple proof (oral communication) of the following fact: if D is a Sobolev coefficient c and local norm zero, the integrability property (iv) holds for all positive α that satisfy $\alpha < c^{-1}$. Consequently we have:

PROPOSITION 5.2. *If D is a Sobolev generator, properties* (i) *to* (v) *of remarks* 5.1 *and* 5.3 *hold*:

Now we prove that hyper and supercontractive properties of the Dirichlet semigroup are intimately connected with the exponential decay of the ground-state eigenfunction of the Schrödinger operator. The sufficiency of condition (5.12) below was already proved and used in the works of J. P. Eckmann, J. Rosen and B. Simon (see [9, Theorem 2.1 and Lemma 2.3], [33, Assumptions of Theorem 1, 3 and 4], [41, Sect. 7]) and our proof mimics that of [33, Theorem 1].

PROPOSITION 5.3. *D is a Sobolev generator with Sobolev coefficient c if and only if we have*:

$$-\operatorname{Log} \psi \leqslant cD + b \qquad\qquad (5.12)$$

as quadratic forms on $L^2(\mu)$ for some constant b.

Proof. By Proposition 5.2 we have $\psi \in L^1$. Thus, if D is a Sobolev generator with Sobolev coefficient c, the function $U = -\operatorname{Log} \psi$ satisfies the assumptions of Lemma 5.2 and (5.12) follows.

Conversely, let us assume that (5.12) is true and let $\epsilon > 0$ be fixed. Because of the homogeneity of (5.1) and because of Remark 5.1 it suffices to prove (5.1) for $f \in C_c^\infty(\mathbb{R}^n)$ such that $\|f\|_{L^2(\mu)} = \epsilon$. Let such an f be fixed and let us set:

$$A = \{x \in \mathbb{R}^n; \; |f(x)| > \psi(x)^{-1}\}.$$

Classical Sobolev's inequality implies:

$$\mathbb{1}_A \operatorname{Log}(\psi \, |f|) \leqslant k_n(-\tfrac{1}{2}\Delta)$$

as quadratic forms on L^2, with k_n equal to $\| \mathbb{1}_A \, \mathrm{Log}(\psi \mid f \mid) \|_{n/2}$ times the constant appearing in classical Sobolev's inequality. Consequently k_n equals ϵ times a constant a_n which depends only on n. Furthermore:

$$\mathrm{Log} \mid f \mid \; \leqslant \; \mathbb{1}_A \, \mathrm{Log}(\psi \mid f \mid) - \mathrm{Log} \, \psi$$
$$\leqslant \epsilon a_n(-\tfrac{1}{2}\Delta) + c(-\tfrac{1}{2}\Delta + V) + b$$
$$\leqslant (c + 2\epsilon a_n)(-\tfrac{1}{2}\Delta + V) + b'$$

as forms on L^2. This latter inequality is equivalent to:

$$\mathrm{Log} \mid f \mid \; \leqslant (c + 2\epsilon a_n)D + b' \tag{5.13}$$

as forms on $L^2(\mu)$, and pluging f in (5.13) shows that D is a Sobolev generator with Sobolev coefficient $c + 2\epsilon a_n$. Letting $\epsilon \to 0$ gives the desired conclusion. ∎

Together with Proposition 5.1, Proposition 5.3 provides us with a device to prove that some Schrödinger operators give rise to hyper or supercontractive Dirichlet semigroups. In order to prove (5.12) it might be convenient to prove

$$-\mathrm{Log} \, \psi(x) \leqslant d_1 V_1(x) + d_2 \tag{5.14}$$

for some real constants $d_1 \geqslant 0$ and d_2 and almost every x in $\mathbb{R}^n$. Consequently [5, Proposition 4.2] can be used to manufacture examples of $V \in \mathscr{V}$ satisfying (5.14) and which could not be captured by the techniques of [9, 33, 41]. We conclude the present paper with another example for which (5.14) is satisfied. More than the particular features of the very example we study, we think that the nature of the proof is instructive and could be proved to be useful in other situations.

PROPOSITION 5.4. *Let V be a potential of class $\mathscr{V}$ which satisfies:*

$$\forall x \in \mathbb{R}^n, \qquad W_x\{\mid X_t \mid_x \leqslant \alpha\}^{-1} \int_0^t \int_{\mid y \mid_\infty \leqslant \alpha} \int_{\mathbb{R}^n} V_1(z) \, p_s(x, z) \, p_{t-s}(z, y) \, dz \, dy \, ds$$
$$\leqslant a' V_1(x) + b' \tag{5.15}$$

for some positive real numbers α, t, a' and b', and where $\mid x \mid_x = \max_{j=1,\dots,n} \mid x_j \mid$ if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then, there is a positive constant a for which:

$$\forall x \in \mathbb{R}^n, \qquad -\mathrm{Log} \, \psi(x) \leqslant a[1 + \mid x \mid^2 + V_1(x)]. \tag{5.16}$$

Proof. By the local boundedness away from zero of ψ, the real number:

$$\epsilon(\alpha) = \inf\{\psi(y); \mid y \mid_x \leqslant \alpha\}$$

is strictly greater than zero. Now, if $x \in \mathbb{R}^n$ is fixed we have:

$$\psi(x) = e^{tE} E_{W_x} \left\{ \psi(X_t) \exp\left[-\int_0^t V_1(X_s)\, ds\right] \exp\left[\int_0^t V_2(X_s)\, ds\right] \right\}$$

$$\geqslant e^{tE} E_{W_x} \left\{ \psi(X_t) \exp\left[-\int_0^t V_1(X_s)\, ds\right]; \ |X_t|_\infty \leqslant \alpha \right\}$$

$$\geqslant e^{tE} \epsilon(\alpha) E_{W_x} \left\{ \exp\left[-\int_0^t V_1(X_s)\, ds\right]; \ |X_t|_\infty \leqslant \alpha \right\}$$

$$\geqslant e^{tE} \epsilon(\alpha) W_x\{|X_t|_\infty \leqslant \alpha\}$$

$$\times \exp\left[-W_x\{|X_t|_\infty \leqslant \alpha\}^{-1} \int_0^t E_{W_x}\{V_1(X_s); \ |X_t|_\infty \leqslant \alpha\}\, ds\right]$$

by Jensen's inequality. Equivalently:

$$-\mathrm{Log}\,\psi(x) \leqslant -tE - \mathrm{Log}\,\epsilon(\alpha) - \mathrm{Log}\,W_x\{|X_t|_\infty \leqslant \alpha\} + W_x\{|X_t|_\infty \leqslant \alpha\}^{-1}$$

$$\times \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} V_1(z)\, \mathbb{1}_{\{|y|_\infty \leqslant \alpha\}}(y)\, p_s(x, z)\, p_{t-s}(z, y)\, dz\, dy\, ds \quad (5.17)$$

(5.17) gives the desired conclusion because of (5.15) and because:

$$-\mathrm{Log}\,W_x\{|X_t|_\infty \leqslant \alpha\} \leqslant a(1 + |x|^2)$$

for some positive constant a. ∎

Remark 5.5. Even though the technique has proved to be very useful in [5], we do not intend to substitute functions of x to t and α. That is the reason why we did not keep track of the dependence of the constants on t.

EXAMPLE. We claim that if V_1 satisfies:

$$\forall x \in \mathbb{R}^n, \qquad a_1 P(x) + b_1 \leqslant V_1(x) \leqslant a_2 P(x) + b_2 \qquad (5.18)$$

for real constants $a_1 > 0$, $a_2 > 0$, b_1 and b_2 and for a polynomial function P of the form:

$$P(x_1, \ldots, x_n) = \sum_{j_1, \ldots, j_n \geqslant 0} a_{j_1, \ldots, j_n} x_1^{2j_1} \cdots x_n^{2j_n} \qquad (5.19)$$

with all the coefficients $a_{j_1, \ldots, j_n}$ non-negative, then, (5.16) is satisfied. The proof is based on the following:

LEMMA 5.4. *For each integer $j \geqslant 0$ there is a polynomial $Q(\alpha, \beta)$ of global degree j, of degree $\mathrm{int}(j/2)$ in β, the only term of degree j being α^j, and such that: for all real numbers x, y, s and t such that $0 < s < t$, if we set:*

$$I_j(x, y) = I_j = \int_{-\infty}^{+\infty} z^j \, e^{-(z-x)^2/2s} \, e^{-(z-y)^2/2(t-s)} \, dz,$$

then we have:

$$I_j = Q_j(st^{-1}y + (t - s) t^{-1}x, s(t - s) t^{-1}) I_0. \tag{5.20}$$

Here $\mathrm{int}(x)$ stands for the integer part of the real number x.

Proof. If $j \geqslant 1$ we have:

$$I_j = \int_{-\infty}^{+\infty} z^{j-1}(z - x) \, e^{-(z-x)^2/2s} \, e^{-(z-y)^2/2(t-s)} \, dz + xI_{j-1}$$

$$= s(j - 1) I_{j-2} - s(t - s)^{-1} I_j + s(t - s)^{-1} I_{j-1} + I_{j-1} \,,$$

by an integration by parts. In fact we have:

$$I_j = st^{-1}(t - s)(j - 1) I_{j-2} + (st^{-1}y + (t - s) t^{-1}x) I_{j-1} \quad \text{if} \quad j \geqslant 2$$

and:

$$I_1 = (st^{-1}y + (t - s) t^{-1}x) I_0 \,.$$

A simple induction argument concludes the proof. ∎

Now, let us assume that V_1 satisfies (5.18) with P given by (5.19), and let us check (5.15).

$$W_x\{|X_t|_\infty \leqslant \alpha\}^{-1} \int_0^t \int_{|y|_\infty \leqslant \alpha} \int_{\mathbb{R}^n} V_1(z) \, p_s(x, z) \, p_{t-s}(z, y) \, dz \, dy \, ds$$

$$= W_x\{|X_t|_\infty \leqslant \alpha\}^{-1} \int_0^t a_2 \sum_{j_1,\ldots,j_n} a_{j_1,\ldots,j_n}(2\pi s)^{-n/2}[2\pi(t - s)]^{-n/2}$$

$$\times \prod_{i=1}^n \int_{-\alpha}^{+\alpha} I_{2j_i}(x_i, y_i) \, dy_i \, ds + b_2$$

$$\leqslant W_x\{|X_t|_\infty \leqslant \alpha\}^{-1} \int_0^t a_2 \sum_{j_1,\ldots,j_n} a_{j_1,\ldots,j_n}(2\pi s)^{-n/2}[2\pi(t - s)]^{-n/2}$$

$$\times \prod_{j=1}^n (c_{2j_i} + x_i^{2j_i}) \int_{-\alpha}^{+\alpha} I_0(x_i, y_i) \, dy_i \, ds + b_2$$

$$\leqslant at[P(x) + 1] \tag{5.21}$$

for some positive constant a, where we used Lemma 5.4, the estimate:

$$\forall x \in \mathbb{R}^n \qquad \sup_{0 \leqslant s \leqslant t} \sup_{y \in \{-\alpha, +\alpha\}} Q_{2j}(st^{-1}y + (t-s)t^{-1}x, s(t-s)t^{-1}) \leqslant c_{2j} + x^{2j}$$

which holds for some positive constant c_{2j}, and the identity:

$$W_x\{|X_t|_\infty \leqslant \alpha\} = (2\pi s)^{-n/2}[2\pi(t-s)]^{-n/2} \prod_{i=1}^{n} \int_{-\alpha}^{+\alpha} I_0(x_i, y_i)\, dy_i$$

which is nothing but the Chapman–Kolmogorov's relation. The conjunction of (5.21) and of (5.18) concludes the proof of the claim.

We summarize the above results in the following:

PROPOSITION 5.5. *Let V be a potential of class $\mathscr{V}$ which satisfies* (5.18) *for a polynomial function P satisfying* (5.19) *and:*

$$\liminf_{|x| \to \infty} |x|^{-2}P(x) > 0.$$

Then the corresponding Dirichlet operator is a Sobolev generator and the associated Dirichlet semigroup is hypercontractive.

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