Large deviations for random walks

Let $(X_n)_{n\geqslant 1}$ be a sequence of iid Bernoulli(p) random variables. Consider the process $S_n = X_1 + \cdots + X_n$ with $S_0 = 0$. Define a continuous random function φ_n on $[0,1]$ by

$$\varphi_n(t) = \frac{S_k}{n} + \frac{(S_{k+1} - S_k)}{n}(t-k) \qquad \text{for } k \leqslant t < k+1.$$

Let $\mathcal{K}$ be the subset of $C([0,1])$ such that $f \in \mathcal{K}$ if and only if $f(0) = 0$ and $|f(t) - f(s)| \leqslant |t-s|$ for all $0 \leqslant s \leqslant t \leqslant 1$. Observe that $\varphi_n \in \mathcal{K}$ and that using the notations of the section "Large deviations for processes" of the 4th lecture notes of the course we have

$$\varphi_n(t) = \int_0^t F_n(X_{\leqslant n})(s)\,\mathrm{d}s$$

where for every n the vectors $X_{\leqslant n} = (X_1, \ldots, X_n)$ are random elements in $\{0,1\}^n$ and

$$F_n(x_1, \ldots, x_n)(\theta) = \sum_{i=1}^n x_i \, 1_{\theta \in [(i-1)/,i/n)} \in L^\infty([0,1]).$$

Observe also that $\varphi_n(t)$ is a piecewise linear function for which

$$\varphi_n(k/n) = S_k/n.$$

Recall that J_p has been defined in Poly 4 (section "Large deviations for processes") as

$$J_p(x) = H(\mathrm{Ber}(x)|\mathrm{Ber}(p)) = x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}$$

the relative entropy of a Bernoulli law of parameter x with respect to the Bernoulli law of parameter p.

a) Prove that the set $\mathcal{K}$ with the norm of uniform convergence is compact.

b) Prove that the laws μ_n of φ_n on $\mathcal{K}$ satisfy a large deviation principle with rate function

$$I(f) = \int_0^1 J_p(f'(s))\mathrm{d}s$$

where $f'(s)$ is the derivative of $f \in \mathcal{K}$ (which exists almost everywhere since f is Lipshitz). Hint: use Theorem 8 (Mogulskii theorem) of Poly 4 and the contraction principle (Thm 19 of Poly 3).

c) With $p = 1/2$, use the result of point b) to prove that, if we set

$$p_n = \mathbb{P}\left(\frac{S_k}{n} \geqslant c\left(\frac{k}{n}\right)^2 \quad \text{for all } k = 1, \ldots, n\right)$$

then

$$(p_n)^{1/n} \to \begin{cases} 1 & \text{for } c \in [0, 1/2] \\ \frac{1}{2c^c(1-c)^{1-c}} & \text{for } c \in [1/2, 1] \end{cases}$$

Hint: use Corollary 9 of Poly 3 to convert the limit computation in a variational problem over the rate function, guess the shape of the extremal function and then use the fact that the function $J_{1/2}$ is convext and Jensen's inequality gives

$$\int_0^1 J_{1/2}(\varphi(s))\mathrm{d}s \geqslant J_{1/2}(\int_0^1 \varphi(s)\mathrm{d}s).$$

To understand fully this problem it is a good idea to give a look at the paper

`http://www.math.tau.ac.il/~tsirel/Courses/LargeDev/lect5.pdf`

in particular Section 5a.