Curie-Weiss model

This is a model of interacting components where it is possible to use large deviation theory to understand the collective behavior of the system. Fix $\beta \geqslant 0$ and let $\Omega_n = \{-1, 1\}^n$ and consider the probability measure P_n over Ω_n given by

$$P_n(\omega_1, \ldots, \omega_n) = \frac{1}{2^n} \frac{e^{\beta H_n(\omega)}}{Z_n}, \qquad \omega = (\omega_1, \ldots, \omega_n) \in \Omega_n$$

where

$$H_n(\omega) = \frac{1}{n} \sum_{i,j=1}^{n} \omega_i \omega_j$$

and where Z_n is a normalization constant such that

$$P_n(\Omega_n) = \sum_{\omega \in \Omega_n} P_n(\omega_1, \ldots, \omega_n) = 1$$

that is

$$Z_n = \sum_{\omega \in \Omega_n} \frac{\exp(\beta H_n(\omega))}{2^n}.$$

Observe that $H_n(\omega) = n\beta F(M_n(\omega))$ where $F(x) = x^2$ and $M_n(\omega) = (\omega_1 + \cdots + \omega_n)/n$ is the empirical mean of the random variables $\omega_1, \ldots, \omega_n = \pm 1$. Let μ_n be the law of M_n under the probability measure P_n. Observe that $M_n: \Omega_n \to [-1, 1]$ so that μ_n is a probability over $[-1, 1]$ for all $n \geqslant 1$ and that the law μ_n is defined by

$$\int_{[-1,1]} \varphi(x) \mu_n(\mathrm{d}x) = \sum_{\omega \in \Omega_n} \varphi(M_n(\omega)) P_n(\omega) = \sum_{\omega \in \Omega_n} \varphi((\omega_1 + \cdots + \omega_n)/n) \frac{\exp(\beta H_n(\omega))}{2^n Z_n}$$

a) Prove that the family $\{\mu_n\}_{n \geqslant 1}$ satisfy a large deviation principle on $[-1, 1]$ with rate function

$$I(x) = \frac{1+x}{2} \log(1+x) + \frac{1-x}{2} \log(1-x) + \beta x^2 - C$$

where the constant C is chosen such that $\inf_{x \in [-1,1]} I(x) = 0$. Hint: Observe that the measure μ_n can be written as

$$\int_{[-1,1]} \varphi(x) \mu_n(\mathrm{d}x) = \int_{[-1,1]} \varphi(x) \frac{e^{\beta n F(x)}}{Z_n} \mu_n^0(\mathrm{d}x)$$

where μ_n^0 is the law of M_n under P_n when $\beta = 0$, that is when each $\omega_1, \ldots, \omega_n$ is an independent random variable such that $P_n(\omega_n = \pm 1) = 1/2$. So that by Cramers theorem $\{\mu_n^0\}_{n \geqslant 1}$ satisfy a large deviation principle with suitable rate function. Use then Theorem 21 (Change of measure) of Poly 3.

b) Prove that for $\beta \in]0, 1]$ the unique minimizer of the function $I: [-1, 1] \to \mathbb{R}_+$ is 0 and that if $\beta > 1$ then there exists a function $m(\beta)$ such that the function I has exaclty two minima at the points $\pm m(\beta)$.

c) Use the above results to prove that, when $\beta \leqslant 1$ the family μ_n weakly converges to the Dirac mass in 0 and that when $\beta > 1$ it weakly converge to the probability measure

$$\frac{1}{2}\delta_{-m(\beta)} + \frac{1}{2}\delta_{m(\beta)}$$

that is the discrete measure that assign probability 1/2 to the points $\pm m(\beta)$. Hint: when $\beta \leqslant 1$ consider the probability $\mu_n(]-\varepsilon, \varepsilon[^c)$ for some $\varepsilon > 0$ and use the large deviation estimate to show that it goes to zero as $n \to \infty$. Use a similar strategy in the case $\beta > 1$.

To understand fully this problem it is a good idea to give a look at the paper

`http://www.math.umass.edu/~rsellis/pdf-files/Les-Houches-lectures.pdf`

in particular Chapter 9.1.