

Noise sensitivity and Social Chaos

As we have already seen, a Bf $f: \{\pm 1\}^n \rightarrow \{\pm 1\}$ can encode a neutral social choice function $F: L_m^n \rightarrow \{\pm 1\}^{n(n-1)/2}$ among m alternatives which satisfy the IIA assumption. Between choice a and b society will prefer a to b if $f(x^{a,b}) = 1$ where $x_i^{a,b} = 1$ if the i -th individual prefer a to b and -1 otherwise. In the following we will assume that f is monotone.

Each Bf f represent a simple voting game where n players (electors) vote to elect some candidate. Each of such games is defined by a function $v: \mathcal{P}([n]) \rightarrow \{0, 1\}$ such that $v(S) = 1$ if $S \subseteq [n]$ is a winning colation of voters (that is if the fact that all element in S vote for a candidate results in his election) otherwise the colation is a losing one. We assume that $v(\emptyset) = 0$ and $v([n]) = 1$ and that $S \subseteq R, v(S) = 1 \Rightarrow v(R) = 1$. The game is proper if $v(S) + v(S^c) \leq 1$ and strong if $v(S) + v(S^c) = 1$. A strong simple game is represented by the Bf $f(x) = 2v(\{i: x_i = 1\}) - 1$ which is monotone and odd. In the following we will consider only these functions.

We write $P_{\text{cyc}}(f)$ for the probability that the neutral IIA social choice function over 3 alternatives (or equivalently the strong simple game G) leads to cyclic social preferences as in the Arrow's theorem. For 3 alternatives we have 8 possible outcomes for the social choice function of which 2 are cyclic. Under our assumptions and uniform individual preference profiles all the non-cyclic and all the cyclic social profiles have the same probability so if we choose a profile at random we have $1/4$ probability to obtain a cyclic result. This motivates the following definition.

Definition 1. We say that a sequence $(f_n)_n$ of strong simple games leads to social chaos iff

$$P_{\text{cyc}}(f_n) \rightarrow 1/4.$$

Recall that Arrow's theorem imply that $P_{\text{cyc}}(f_n) > 0$ unless f_n is a dictatorship and that if f_n is the simple majority function then, as we have already seen,

$$P_{\text{cyc}}(f_n) = \frac{1}{4} + \frac{3}{2\pi} \arcsin\left(\frac{1}{3}\right) \simeq 0.08744.$$

A remarkable observation is that this notion of social chaos is tightly linked with the notion of noise-sensitivity. Let $N_\rho(f) = \mathbb{P}_\rho(f(x) \neq f(y))$ where under \mathbb{P}_ρ the vector x is uniformly drawn over $\{\pm 1\}^n$ and $y_i = \pm x_i$ with probability $(1 \pm \rho)/2$ independently for each coordinate $i = 1, \dots, n$. The value $N_\rho(f)$ is the probability that random errors will affect the outcome of the election ruled by f .

Definition 2. We say that a sequence $(f_n)_n$ of strong simple games is asymptotically noise-sensitive iff $\forall 0 \leq \rho < 1$,

$$N_\rho(f_n) \rightarrow 1/2.$$

Here $1/2$ corresponds to the case that $f(x)$ and $f(y)$ are drawn at random independently: the errors affecting the counting of the votes of each individual induce a result $f(y)$ for the election which has nothing to do with the "real" outcome $f(x)$. Observing $f(y)$ does not say anything on $f(x)$.

Theorem 3. (Kalai) *Social chaos is equivalent to noise-sensitivity.*

Proof. The function f_n is odd so $W_k(f_n) = 0$ for all even $n \geq 0$. Recall the Fourier representation for $P_{\text{cyc}}(f_n)$:

$$\begin{aligned} P_{\text{cyc}}(f_n) &= 1 - P(\text{Rational}(f)) = 1 - \frac{3}{4} + \frac{3}{4} \sum_{S \subseteq [n]} \left(-\frac{1}{3}\right)^{|S|} \hat{f}(S)^2 \\ &= \frac{1}{4} - \frac{1}{4} \sum_{k=1}^n \left(\frac{1}{3}\right)^{k-1} W_k(f_n) \end{aligned}$$

since there are contributions only from k odd. On the other hand

$$\begin{aligned} N_\rho(f_n) &= \mathbb{P}_\rho(f_n(x) \neq f_n(y)) = \mathbb{E}_\rho\left(\frac{1 - f_n(x)f_n(y)}{2}\right) = \frac{1}{2} - \frac{1}{2} \sum_{S \subseteq [n]} \hat{f}_n(S)^2 \rho^{|S|} \\ &= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^n \rho^k W_k(f_n) = \frac{1}{2} - \frac{1}{2} \sum_{k=1}^n \rho^k W_k(f_n). \end{aligned}$$

since $W_0(f_n) = 0$. Then

$$2P_{\text{cyc}}(f_n) + 1 = 3N_{1/3}(f_n)$$

which implies that if $N_{1/3}(f_n) \rightarrow 1/2$ then $P_{\text{cyc}}(f_n) \rightarrow 1/4$. Now assume that $P_{\text{cyc}}(f_n) \rightarrow 1/4$, then

$$0 \leq \sum_{k=1}^n \left(\frac{1}{3}\right)^k W_k(f_n) \rightarrow 0$$

which means that for each k we have $W_k(f_n) \rightarrow 0$. Now note that $\sum_{k=1}^n W_k(f_n) = 1$ so for any $\ell < n$

$$\begin{aligned} \sum_{k=1}^n \rho^k W_k(f_n) &= \sum_{k=1}^{\ell} \rho^k W_k(f_n) + \sum_{k=\ell+1}^n \rho^k W_k(f_n) \\ &\leq \sum_{k=1}^{\ell} W_k(f_n) + \rho^{\ell+1} \rightarrow \rho^{\ell+1} \end{aligned}$$

since each $W_k(f_n) \rightarrow 0$. Given that ℓ is arbitrary we obtain $\sum_{k=1}^n \rho^k W_k(f_n) \rightarrow 0$ as required. \square

Let us now discuss the relation of noise sensitivity with the Banzhaf power index. The Banzhaf power $b_i(G)$ of the i -th individual in the strong simple game G is defined as the probability that this individual is pivotal, that is for a random coalition S not containing i it is the case that S is a losing coalition while $S \cup \{i\}$ is a winning one, namely that i can make a difference. Let $b_{\max}(G) = \max_i b_i(G)$ and $b_{\min}(G) = \min_i b_i(G)$ the maximum and minimum Banzhaf power of the individuals in the game G and let $I(G) = \sum_i b_i(G)$.

Exercise 1. Show that $I(G)$ is the influence (or energy) $\mathcal{E}(f)$ of the corresponding Bf f .

We already know that $I(G)$ is maximized by the simple majority game, indeed for monotone Bf f we have

$$\begin{aligned} D_i f &= f(x_1, \dots, x_{i-1}, +1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_n) \\ &= 2 \sum_{S \ni i} \hat{f}(S) x_{S \setminus \{i\}} \in \{0, 2\} \\ \text{Inf}_i(f) &= \mathbb{P}(D_i f(x) = 2) = \frac{1}{2} \mathbb{E}[D_i f(x)] = 2 \hat{f}(\{i\}). \\ \mathcal{E}(f) &= 2 \sum_i \hat{f}(\{i\}) = 2W_1(f) \end{aligned}$$

and we have already seen that $W_1(f)$ is maximized by the majority function and that

$$W_1(\text{Maj}_n) = \sqrt{\frac{2}{\pi}} n + o(\sqrt{n}).$$

Theorem 4. *If a sequence of strong simple games $(G_n)_n$ leads to social chaos then it has diminishing individual Banzhaf power, i.e.*

$$b_{\max}(G_n) \rightarrow 0.$$

Proof. Assume that $b_{\max}(G_n)$ stay bounded away from zero (maybe take a subsequence so that this happens). Then the outcome of the game has substantial (i.e. bounded away from zero) correlation with one of the influential players which has substantial probability of not being affected by the noise. So the correlation of the outcomes before and after the action of the noise will be equally bounded away from zero contradicting the hypothesis of social chaos (which we showed equivalent to noise sensitivity). \square

A sequence of games has a bounded power ration if $b_{\max}(G_n)/b_{\min}(G_n)$ is uniformly bounded.

Benjamini-Kalai-Schramm proved the following very important characterization of noise-sensitive functions

Theorem 5. (Benjamini-Kalai-Schramm) *The sequence $(f_n)_n$ of Bf is noise-sensitive iff*

$$\sum_i (\text{Inf}_i(f_n))^2 \rightarrow 0.$$

From this result we can infer that

Theorem 6. *The sequence of strong simple games $(G_n)_n$ with bounded power ratio leads to social chaos iff*

$$I(G_n)/\sqrt{n} \rightarrow 0.$$

Proof. Note that

$$\frac{I(G_n)^2}{n} \leq \text{Jensen} \sum_i (\text{Inf}_i(f_n))^2 \leq b_{\max}(G_n)^2 n = \left(\frac{b_{\max}(G_n)}{b_{\min}(G_n)} \frac{I(G_n)}{\sqrt{n}} \right)^2$$

so the claim is a direct consequence of the BKS theorem. \square

Another source of social chaos is given by bias.

Theorem 7. *If $\mathbb{E}[f_n] \rightarrow \pm 1$ then the sequence of strong simple games G_n is noise-sensitive.*

Proof.

□

An example of social chaos

An example of a social welfare function which exhibit social chaos is given by a two-level voting method where the lower level is biased due to supermajority. Assume we have a b voters divided into a communities of b voters. Given subset S and a community C we call C positive if $|S \cap C| \geq t$ and negative if $|S \cap C| \geq b - t$. The subset S is a winning coalition if there are more positive than negative communities. Let $G_n = G_n(a_n, b_n, t_n)$ a sequence of such games. Then we can choose t_n in such a way that the probability p_n that a community is positive goes to zero as $n \rightarrow \infty$.

We have the following

Theorem 8. *Assume that $b_n \rightarrow \infty$ and that ... then the sequence G_n leads to social chaos.*

Proof.

□