Bachelor topics SS2019

✻

Inverting the signature of a path

▷ Lyons, Terry, and Weijun Xu. 2014. "Inverting the Signature of a Path." *ArXiv:1406.7833 [Math]*, June. http://arxiv.org/abs/1406.7833.

▷ Lyons, Terry, and Weijun Xu. 2015. "Hyperbolic Development and Inversion of Signature." *ArXiv:1507.00286 [Math]*, July. http://arxiv.org/abs/1507.00286.

The signature $S(\gamma)$ of a C^1 path $\gamma\colon [0,1]\to\mathbb{R}^d$ consists of the infinite sequence of iterated integrals of the form

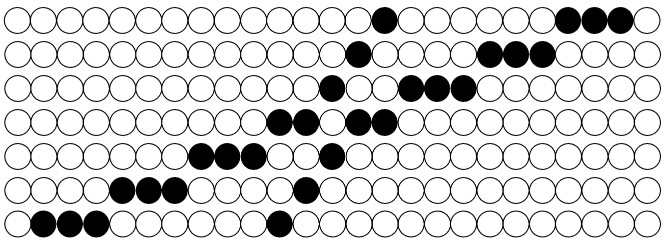
$$\int_{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant 1}\dot{\gamma}_{t_1}\otimes\cdots\otimes\dot{\gamma}_{t_n}\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

It is some kind of nonlinear encoding of the path. In the classical work of K. T. Chen it is proven that the map $\gamma\mapsto S(\gamma)$ is a homomorphism from the monoid of paths with concatenation to the tensor algebra over $\mathbb{R}^d$, and that two piecewise regular paths without backtracks have the same signatures if and only if they differ by a reparametrization and translation. The suggested papers prove that it is possible to effectively reconstruct a path from its signature, giving insights on how the properties of the path are encoded in the signature. Signature has become in recent years an effective tool to encode geometry of paths in ways suitable to machine learning algorithms.

✻

Dynamics of the Ball–Box system with random initial condition

▷ Croydon, David A., Tsuyoshi Kato, Makiko Sasada, and Satoshi Tsujimoto. 2018. "Dynamics of the Box-Ball System with Random Initial Conditions via Pitman's Transformation," June. https://arxiv.org/abs/1806.02147.



The box-ball system (BBS), introduced by Takahashi and Satsuma in 1990, is a cellular automaton that exhibits solitonic behaviour. In this article, the BBS is studied when started from a random two-sided infinite particle configuration. For such a model, Ferrari et al. recently showed the invariance in distribution of Bernoulli product measures with density strictly less than $\frac{1}{2}$, and gave a soliton decomposition for invariant measures more generally. The BBS dynamics is studied using the transformation of a nearest neighbour path encoding of the particle configuration given by 'reflection in the past maximum'. The paper characterise the set of configurations for which the dynamics are well-defined and reversible for all times and gives simple sufficient conditions for random initial conditions to be invariant in distribution under the BBS dynamics. Furthermore, various probabilistic properties of the BBS that are commonly studied for interacting particle systems are studied, such as the asymptotic behavior of the integrated current of particles and of a tagged particle. Finally, for Bernoulli product measures with parameter $p\nearrow\frac{1}{2}$ (which may be considered the 'high density' regime), the path encoding has a natural scaling limit, which motivates the introduction of a new continuous version of the BBS.

✻

Regularity of almost every function in Sobolev spaces

▷ Hunt, Brian R. 1994. "The Prevalence of Continuous Nowhere Differentiable Functions." *Proceedings of the American Mathematical Society* 122 (3): 711–17. https://doi.org/10.1090/S0002-9939-1994-1260170-X.

▷ Fraysse, Aurélia, and Stéphane Jaffard. 2006. "How Smooth Is Almost Every Function in a Sobolev Space?" *Revista Matemática Iberoamericana*, 663–82. https://doi.org/10.4171/RMI/469.

▷ Fraysse, A. 2010. "Regularity Criteria for Almost Every Function in Sobolev Spaces." *Journal of Functional Analysis* 258 (6): 1806–21. https://doi.org/10.1016/j.jfa.2009.11.017.

In the space of continuous functions of a real variable, the set of nowhere differentiable functions has long been known to be topologically "generic". In these papers it is shown further that in a measure theoretic sense (which is different from Wiener measure), "almost every" continuous function belonging to given functional spaces (like Holder or Sobolev) is as irregular as it can be. Let us give an example: recall that $f \in C^{\alpha}(x_0)$ if there exists $C < \infty$ such that

$$|f(x) - P(x - x_0)| \leqslant C|x - x_0|^{\alpha}$$

for $|x - x_0| \leqslant 1$. Then one can prove that for "almost every" $f \in C_0^s(\mathbb{R}^d)$ one has that $f \in C^{\alpha}(x_0)$ for all x_0 and all $\alpha \leqslant s$ but not $\alpha > s$. When considering Soboles spaces there will be different regularities in different sets of points, namely a generic function possess a "multifractal" spectrum.

✻

Branched polymers and dimensional reduction

▷ Brydges, David, and John Imbrie. 2003. "Branched Polymers and Dimensional Reduction." *Annals of Mathematics* 158 (3): 1019–39. https://doi.org/10.4007/annals.2003.158.1019.

This paper establish an exact relation between self-avoiding branched polymers in $D+2$ continuum dimensions and the hard-core continuum gas at negative activity in D dimensions.

Hard core gas in D dimensions.

$$Z_{\mathrm{HC}}(z) = \sum_{N \geqslant 0} \frac{z^N}{N!} \int_{\Lambda^N} (\mathrm{d}^D x)^N \prod_{i<j} \mathbb{I}_{|x_i - x_j| \geqslant 1}.$$

Branched polymer in $D+2$ dimensions.

$$Z_{\mathrm{BP}}(z) = \sum_{N \geqslant 1} \frac{z^N}{N!} \sum_{T \text{ tree on } 1,\ldots,N} \int \prod_{ij \in T} \underbrace{\mathrm{d}\Omega(y_{ij})}_{\substack{\text{surface measure}\\ \text{on unit ball}}} \prod_{ij \notin T} \mathbb{I}_{|y_{ij}| \geqslant 1}.$$

Then

$$\frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(z) \to -2\pi Z_{\mathrm{BP}}\left(-\frac{z}{2\pi}\right).$$

This can be used when $D = 0, 1$ to compute critical exponents for the branched polymer. The scope of the thesis is to understand the proof of the equivalence and some of the rigorous consequences.

✻

Game–theoretic probability

▷ Shafer, Glenn, and Vladimir Vovk. 2001. *Probability and Finance: It's Only a Game!* 1 edition. New York: Wiley-Interscience.

The aim of the thesis is to illustrate the proof of some central theorems of probability (like the law of large numbers, the law of iterated logarithm, etc⟨text-dots⟩) in the game theoretic framework presented in the above book. There probability theory is developed, not starting from measure theory, as usual, but by considering betting systems between Nature and Skeptik whose "experiments" put to a test the hypothesised randomness of nature. Randomness is therefore defined as the impossibility of certain outcomes for these games. Surprisingly this approach can go very far and gives a different perspective to the philosophical aspects of a mathematical theory of randomness.

✻

Probabilistic representation for Navier–Stokes equation

▷ Ossiander, Mina. 2005. "A Probabilistic Representation of Solutions of the Incompressible Navier-Stokes Equations in R3." *Probability Theory and Related Fields* 133 (2): 267–98. https://doi.org/10.1007/s00440-004-0418-z.

The paper introduces a probabilistic representation for solutions of the incompressible Navier-Stokes equations in $\mathbb{R}^3$ with given forcing and initial velocity. This representation expresses solutions as scaled conditional expectations of functionals of a Markov process indexed by the nodes of a binary tree. It gives existence and uniqueness of weak solutions for all time under relatively simple conditions on the forcing and initial data. These conditions involve comparison of the forcing and initial data with majorizing kernels. It constitute an interesting use of probability theory to represent solutions to certain nonlinear PDEs.

✻

Onsager's conjecture and weak solutions of Euler's equation

▷ De Lellis, Camillo, and László Székelyhidi Jr. "Dissipative Euler Flows and Onsager's Conjecture." ArXiv e-print, May 16, 2012. http://arxiv.org/abs/1205.3626.

Onsager's conjecture states that energy conservation for weak solutions of Euler's equation for the motion of an inviscid perfect fluid

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

is linked with the Hölder regularity of the solutions. Namely if $|u(t, x) - u(t, y)| \lesssim |x - y|^\theta$ for $\theta > 1/3$ then energy is conserved and that there exists weak solutions with $\theta < 1/3$ which violates energy conservation. This issue is important in our understanding of weak solutions and in general of irregular solutions to fluid dynamical equations and also for discussion of turbulence. The above paper provide a partial confirmation of the negative side of Onsager's conjecture. Objective of the thesis is the investigation of weak solutions, conservation of energy with enough regularity and describe the construction of counterexamples given in the above paper.

✻✻✻