Bachelor topics SS2020

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Signature of a path

The signature $S(\gamma)$ of a C^1 path $\gamma:[0,1]\to\mathbb{R}^d$ consists of the infinite sequence of iterated integrals of the form

$$\int_{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant 1}\dot{\gamma}_{t_1}\otimes\cdots\otimes\dot{\gamma}_{t_n}\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

It is a kind of nonlinear encoding of the path. In the classical work of K. T. Chen it is proven that the map $\gamma\mapsto S(\gamma)$ is a homomorphism from the monoid of paths with concatenation to the tensor algebra over $\mathbb{R}^d$, and that two piecewise regular paths without backtracks have the same signatures if and only if they differ by a reparametrization and translation. Several possile bachelor thesis topics are grounded in this concept.

a) *Inversion of the signature.* It is possible to effectively reconstruct a path from its signature, giving insights on how the properties of the path are encoded in the signature. Signature has become in recent years an effective tool to encode geometry of paths in ways suitable to machine learning algorithms.

   ▷ Lyons, Terry, and Weijun Xu. 2014. “Inverting the Signature of a Path.” *ArXiv:1406.7833 [Math]*, June. http://arxiv.org/abs/1406.7833.

   ▷ Lyons, Terry, and Weijun Xu. 2015. “Hyperbolic Development and Inversion of Signature.” *ArXiv:1507.00286 [Math]*, July. http://arxiv.org/abs/1507.00286.

b) *Signature of a random path.* The normalized sequence of moments characterizes the law of any finite-dimensional random variable. The first paper proves an analogous result for path-valued random variables, that is stochastic processes, by using the normalized sequence of signature moments. We use this to define a metric for laws of stochastic processes. This metric can be efficiently estimated from finite samples, even if the stochastic processes themselves evolve in high-dimensional state spaces. Possible application is a non-parametric two-sample hypothesis test for laws of stochastic processes. The second paper continues the study of signature as characteristic of stochastic processes looking at comulants and at a new characterisation of independence of multivariate stochastic processes. As an application it obtains a family of unbiased minimum-variance estimators of signature cumulant.

   ▷ Chevyrev, I., 2013. A Set of Characteristic Functions on the Space of Signatures. *ArXiv:1307.3580*, 2013. http://arxiv.org/abs/1307.3580.

   ▷ Chevyrev, Ilya, and Harald Oberhauser. Signature Moments to Characterize Laws of Stochastic Processes. *ArXiv:1810.10971*, 2018. http://arxiv.org/abs/1810.10971.

   ▷ Bonnier, Patric, and Harald Oberhauser. Signature Cumulants, Ordered Partitions, and Independence of Stochastic Processes. *ArXiv:1908.06496*, 2019. http://arxiv.org/abs/1908.06496.

c) *Applications to machine learning.* The specific algebraic properties of the signature make it a powerful method of feature extraction for machine learning purposes.

   ▷ Király, F.J., Oberhauser, H., 2016. Kernels for sequentially ordered data. arXiv:1601.08169. http://arxiv.org/abs/1601.08169.

▷ Lyons, T., Oberhauser, H., 2017. Sketching the order of events. arXiv:1708.09708. http://arxiv.org/abs/1708.09708

▷ Chevyrev, I., Kormilitzin, A., 2016. A Primer on the Signature Method in Machine Learning. arXiv:1603.03788. http://arxiv.org/abs/1603.03788

▷ Chevyrev, I., Nanda, V., Oberhauser, H., 2018. Persistence paths and signature features in topological data analysis. IEEE Transactions on Pattern Analysis and Machine Intelligence 1–1. https://doi.org/10.1109/TPAMI.2018.2885516

▷ Toth, C., Oberhauser, H., 2019. Variational Gaussian Processes with Signature Covariances. arXiv:1906.08215. http://arxiv.org/abs/1906.08215

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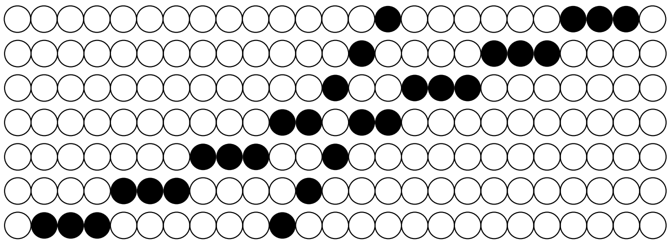
Dynamics of the Ball–Box system with random initial condition



The box-ball system (BBS), introduced by Takahashi and Satsuma in 1990, is a cellular automaton that exhibits solitonic behaviour. In this article, the BBS is studied when started from a random two-sided infinite particle configuration. For such a model, Ferrari et al. recently showed the invariance in distribution of Bernoulli product measures with density strictly less than $\frac{1}{2}$, and gave a soliton decomposition for invariant measures more generally. The BBS dynamics is studied using the transformation of a nearest neighbour path encoding of the particle configuration given by `reflection in the past maximum'. The paper characterise the set of configurations for which the dynamics are well-defined and reversible for all times and gives simple sufficient conditions for random initial conditions to be invariant in distribution under the BBS dynamics. Furthermore, various probabilistic properties of the BBS that are commonly studied for interacting particle systems are studied, such as the asymptotic behavior of the integrated current of particles and of a tagged particle. Finally, for Bernoulli product measures with parameter $p \nearrow \frac{1}{2}$ (which may be considered the `high density' regime), the path encoding has a natural scaling limit, which motivates the introduction of a new continuous version of the BBS.

▷ Croydon, David A., Tsuyoshi Kato, Makiko Sasada, and Satoshi Tsujimoto. 2018. “Dynamics of the Box-Ball System with Random Initial Conditions via Pitman's Transformation,” June. https://arxiv.org/abs/1806.02147.

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Regularity of almost every function in Sobolev spaces

In the space of continuous functions of a real variable, the set of nowhere differentiable functions has long been known to be topologically ``generic”. In these papers it is shown further that in a measure theoretic sense (which is different from Wiener measure), ``almost every'' continuous function belonging to given functional spaces (like Holder or Sobolev) is as irregular as it can be. Let us give an example: recall that $f \in C^{\alpha}(x_0)$ if there exists $C < \infty$ such that

$$|f(x) - P(x - x_0)| \leqslant C|x - x_0|^{\alpha}$$

for $|x-x_0| \leqslant 1$. Then one can prove that for "almost every" $f \in C_0^s(\mathbb{R}^d)$ one has that $f \in C^\alpha(x_0)$ for all x_0 and all $\alpha \leqslant s$ but not $\alpha > s$. When considering Soboles spaces there will be different regularities in different sets of points, namely a generic function possess a "multifractal" spectrum.

▷ Hunt, Brian R. 1994. "The Prevalence of Continuous Nowhere Differentiable Functions." *Proceedings of the American Mathematical Society* 122 (3): 711–17. https://doi.org/10.1090/S0002-9939-1994-1260170-X.

▷ Fraysse, Aurélia, and Stéphane Jaffard. 2006. "How Smooth Is Almost Every Function in a Sobolev Space?" *Revista Matemática Iberoamericana*, 663–82. https://doi.org/10.4171/RMI/469.

▷ Fraysse, A. 2010. "Regularity Criteria for Almost Every Function in Sobolev Spaces." *Journal of Functional Analysis* 258 (6): 1806–21. https://doi.org/10.1016/j.jfa.2009.11.017.

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Reciprocal processes

▷ Léonard, Christian; Rœlly, Sylvie; Zambrini, Jean-Claude, Reciprocal processes. A measure-theoretical point of view *Probab. Surv.* 11 (2014), 237–269. https://doi.org/10.1214/13-PS220

The bridges of a Markov process are also Markov. But an arbitrary mixture of these bridges fails to be Markov in general. However, it still enjoys the interesting properties of a reciprocal process. The paper reviews the main properties of reciprocal processes with emphasis on their time-symmetries. It follows a measure-theoretical approach which allows for a unified treatment of the diffusion and jump processes. Reciprocal processes are related to solutions of entropy minimizing problems on path space and where inspired by the pioneering work of Schrödinger on the time reversal of quantum dynamics.

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Projections of SDEs onto Submanifolds

In the course "Foundations of stochastic analysis" we described SDE with values on the Euclidean space. A more general notion of SDE is that taking values on an arbitrary smooth manifold which requires a slightly different point of view. This thesis will investigate the problem of defining a notion of projection of stochastic process unto a submanifold via stochastic calculus. There is not a unique solution and several possibilities are explored.

▷ J. Armstrong, D. Brigo, E.R. Ferrucci, **Projections of SDEs onto Submanifolds**- arXiv preprint arXiv:1810.03923, 2018. https://arxiv.org/abs/1810.03923

For a general introduction to SDE on manifold see

▷ Hsu, Elton P. **Stochastic analysis on manifolds.** Graduate Studies in Mathematics, 38. *American Mathematical Society, Providence, RI,* 2002. [Chapter 1 and 2]

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Derivation of thermodynamic laws from microscopic stochastic models

The laws of thermodynamics rules behaviour of systems which have a very large number of microscopic degrees of freedom in statistical equilibrium. In particular it formalises the concept of heat and of properties like "warmer" and "colder". The suggested literature derives rigorously thermodynamic behaviour from a system of coupled SDE as the number of components is sent to infinity. This thesis will allow to grasp concepts like "energy", "entropy", "adiabatic transformations", etc… within a rigorous mathematical model of thermodynamics.

▷ Olla, S., 2013. Microscopic Derivation of an Isothermal Thermodynamic Transformation. arXiv:1310.0798. https://arxiv.org/abs/1310.0798

▷ Olla, S., Simon, M., 2015. Microscopic derivation of an adiabatic thermodynamic transformation. Braz. J. Probab. Stat. 29, 540–564. https://doi.org/10.1214/14-BJPS275

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Finite dimensional filters

▷ Cohen de Lara, M., **Finite-dimensional filters. I. The Wei-Norman technique.**, *SIAM J. Control Optim.* 35 (1997), no. 3, 980–1001.

▷ Cohen de Lara, M., **Finite-dimensional filters. II. Invariance group techniques.**, *SIAM J. Control Optim.* 35 (1997), no. 3, 1002–1029.

For an introduction to the theory of stochastic filtering see

▷ Bain, Alan; Crisan, Dan, **Fundamentals of stochastic filtering.**, Stochastic Modelling and Applied Probability, 60. *Springer, New York,* 2009. [Chapters 2 and 3 (and also 4)]

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Game–theoretic probability (taken)

▷ Shafer, Glenn, and Vladimir Vovk. 2001. *Probability and Finance: It's Only a Game!* 1 edition. New York: Wiley-Interscience.

The aim of the thesis is to illustrate the proof of some central theorems of probability (like the law of large numbers, the law of iterated logarithm, etc...) in the game theoretic framework presented in the above book. There probability theory is developed, not starting from measure theory, as usual, but by considering betting systems between Nature and Skeptik whose “experiments” put to a test the hypothesised randomness of nature. Randomness is therefore defined as the impossibility of certain outcomes for these games. Surprisingly this approach can go very far and gives a different perspective to the philosophical aspects of a mathematical theory of randomness.

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