Note 1

Brownian motion.

1 Definition and equivalent characterizations

Definition 1. *A stochastic process* $B_\cdot : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ *is a Brownian motion iff*

1. $B_0 = 0$,
2. *for any* $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \in \mathbb{R}_+$ *we have that* $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent random variables and* $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$,
3. *for almost every* $\omega \in \Omega$ *the function* $t \mapsto B_t(\omega)$ *is continuous (i.e., in* $C(\mathbb{R}_+, \mathbb{R})$*).*

1.1 Brownian motion as a Gaussian process

Recall the definition of Gaussian process indexed by an arbitrary set I.

Definition 2. $X_I = (X_\alpha)_{\alpha \in I}$ *is a Gaussian process indexed by* I *iff for all* $J \subseteq I$ *finite and* $(\lambda_\alpha)_{\alpha \in J} \subseteq \mathbb{R}$ *we have that the linear combination* $Z = \sum_{\alpha \in J} \lambda_\alpha X_\alpha$ *is a Gaussian random variable. If* I *is finite we also say that the r.v.* X_I *with values in* $\mathbb{R}^I$ *is a Gaussian vector.*

Proposition 3. *The law of a Gaussian process* $(X_\alpha)_{\alpha \in I}$ *is determined by its mean and covariance functions, i.e. by* $\mu_\alpha = \mathbb{E}[X_\alpha]$ *and* $C_{\alpha,\beta} = \mathrm{Cov}(X_\alpha, X_\beta)$ *for* $\alpha, \beta \in I$*. Conversely, for any vector* $\mu = (\mu_\alpha)_{\alpha \in I}$ *and any positive semi-definite matrix* $C = (C_{\alpha,\beta})_{\alpha,\beta \in I}$ *there exist a Gaussian process* $(X_\alpha)_{\alpha \in I}$ *with mean* μ *and covariance* C*.*

Proof. By definition, for $J \subseteq I$ finite and $(\lambda_\alpha)_{\alpha \in J} \subseteq \mathbb{R}$ we have that $Z = \sum_{\alpha \in J} \lambda_\alpha X_\alpha$ is a Gaussian, therefore its caracteristic function in 1 reads:

$$\mathbb{E}[e^{iZ}] = e^{i\mathbb{E}[Z] - \frac{1}{2}\mathrm{Var}(Z)}.$$

But now $\mathbb{E}[Z] = \sum_\alpha \lambda_\alpha \mathbb{E}[X_\alpha]$ and similarly for $\mathrm{Var}(Z) = \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta C_{\alpha,\beta}$. From this we conclude that the caracteristic function of the vector $X_J = (X_\alpha)_{\alpha \in J}$ is given by

$$\varphi_{X_J}((\lambda_\alpha)_{\alpha \in J}) = \mathbb{E}\left[\exp\left(i \sum_{\alpha \in J} \lambda_\alpha X_\alpha\right)\right] = \exp\left(i \sum_\alpha \lambda_\alpha \mathbb{E}[X_\alpha] - \frac{1}{2} \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta C_{\alpha,\beta}\right)$$

and therefore the law of X_J is determined. Since all the finite dimensional marginals are fixed, this means that the law of the process $(X_\alpha)_{\alpha \in I}$ on $\mathbb{R}^I$ is unique.

In order to prove the existence of a Gaussian process with mean μ and covariance C we observe that for each $J \subseteq I$ finite we can construct the law μ^J on $\mathbb{R}^J$ given by the Gaussian with mean vector $\mu^J = (\mu_\alpha)_{\alpha \in J} \in \mathbb{R}^J$ and covariance matrix $C^J = (C_{\alpha,\beta})_{\alpha,\beta \in J} \in \mathbb{R}^{J\times J}$. Moreover for $J' \subseteq J$ we have that $\mu^{J'}$ is the marginal of μ^J wrt. the projection $\pi^{J,J'}: \mathbb{R}^J \to \mathbb{R}^{J'}$. Therefore the family $(\mu^J)_{J \subseteq I, J \text{ finite}}$ is a consisten family of probability law and by Daniell-Kolmogorov theorem there exists a unique measure μ on $\mathbb{R}^I$ for which the $(\mu^J)_J$ are marginals and for which the canonical process on $\mathbb{R}^I$ is indeed a Gaussian process with mean μ and covariance C. □

Theorem 4. *Brownian motion is a continuous Gaussian process on $\mathbb{R}_+$ such that $B_0 = 0$ and*

$$\mathbb{E}[B_t] = 0, \tag{1}$$

$$\operatorname{cov}(B_t, B_s) = \min(t, s). \tag{2}$$

Proof. Fix $n \geqslant 1$ and $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and consider $Z = \sum_{k=1}^n \lambda_k B_{t_k}$ for $(\lambda_k)_{k=1,\ldots,n} \subseteq \mathbb{R}$. It is straighforward that there exists $(\mu_k)_k$ such that $\sum_{k=1}^n \lambda_k B_{t_k} = \sum_{k=1}^n \mu_k (B_{t_k} - B_{t_{k-1}})$, by definition of Brownian motion we have then

$$\mathbb{E}[e^{i\theta Z}] = \mathbb{E}\big[e^{i\sum_{k=1}^n \theta \mu_k (B_{t_k} - B_{t_{k-1}})}\big] = \prod_{k=1}^n \mathbb{E}[e^{i\theta \mu_k (B_{t_k} - B_{t_{k-1}})}] = \prod_{k=1}^n e^{-\frac{1}{2}\theta^2 \mu_k^2 (t_k - t_{k-1})} = e^{-\frac{1}{2}\theta^2 A}$$

with $A = \sum_k \mu_k^2 (t_k - t_{k-1})$. From this computation we deduce that Z is a mean zero Gaussian variable with variance A, so $\operatorname{Var}(Z) = A$. Since this is true for any choice of $(\lambda_k)_k$ and times $(t_k)_k$ we have proven that $(B_t)_{t \geqslant 0}$ is a Gaussian process. The explicit form of the covariance is left as an easy exercise. □

Remark 5. The proof above consist essentially in observing that a vector of independent Gaussian random variables is also a Gaussian vector. Note also that if the Gaussians are not independent then taken together they could fail to form a Gaussian vector (give an example).

Corollary 6. *Let B_t be a continuous Gaussian process with mean (1) and co-variance (2), and suppose that $B_0 = 0$ then B_t is a Brownian motion.*

Proof. We have only to prove that $(B_t)_t$ satisfies the second property of Definition 1. Since $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are Gaussian random variables (being linear combinations of jointly Gaussian random variables) we have to prove that $\operatorname{cov}(B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}}) = 0$ if $i \neq j$. Suppose that $t_j < t_i$ then

$$\begin{aligned}\operatorname{cov}(B_{t_i} - B_{t_{i-1}}, B_{t_j} - B_{t_{j-1}}) &= \operatorname{cov}(B_{t_i}, B_{t_j}) - \operatorname{cov}(B_{t_{i-1}}, B_{t_j}) - \operatorname{cov}(B_{t_i}, B_{t_{j-1}}) + \operatorname{cov}(B_{t_{i-1}}, B_{t_j}) \\ &= t_j - t_j - t_{j-1} + t_{j-1} = 0,\end{aligned}$$

which concludes the proof. □

1.2 Brownian motion as a Markov process

We consider the natural filtration $(\mathscr{F}_t)_{t \geqslant 0}$ of $(B_t)_{t \geqslant 0}$ given by $\mathscr{F}_t = \sigma(B_s, s \in [0, t])$.

Theorem 7. *A Brownian motion B_t is a homogeneous Markov process wrt. $(\mathscr{F}_t)_t$ with transition kernel P_t given by*

$$(P_t f)(x) = \int_{\mathbb{R}} p_t(x-y) f(y) \mathrm{d}y$$

for all bounded and measurable $f: \mathbb{R} \to \mathbb{R}$ where

$$p_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right), \qquad t > 0, x \in \mathbb{R}. \tag{3}$$

That is, for every bounded and measurable $f: \mathbb{R} \to \mathbb{R}$ and $0 \leqslant s < t$,

$$\mathbb{E}[f(B_t)|\mathscr{F}_s] = \mathbb{E}[f(B_t)|B_s] = (P_t f)(B_s).$$

Proof. We have to prove that for any $0 \leqslant s < t$ and any Borel set $A \subset \mathbb{R}$ there exists a version of $\mathbb{P}(B_t \in A|\mathscr{F}_s)$ which is $\sigma(B_s)$ measurable.

By Definition 1 we have that $B_t - B_s$ is independent of $B_s - B_0 = B_s$ and $B_t - B_s \sim N(0, t-s)$, so using the properties of conditional expectation, we have, letting $h(x) = \mathbb{E}[f(Z + x)]$ with $Z \sim N(0, t-s)$:

$$\mathbb{E}[f(B_t)|\mathscr{F}_s] = \mathbb{E}[f(B_t - B_s + B_s)|\mathscr{F}_s] = \mathbb{E}[h(B_s)|\mathscr{F}_s] = \mathbb{E}[h(B_s)|B_s] = h(B_s).$$

Taking the conditional expectation wrt. B_s of this equality we get $h(B_s) = \mathbb{E}[\mathbb{E}[f(B_t)|\mathscr{F}_s]|B_s] = \mathbb{E}[f(B_t)|B_s]$ and observing that $h(x) = P_t f(x)$ we conclude. □

Corollary 8. *For any $0 < t_1 < t_2 < \cdots < t_n$ we have that the law of $(B_{t_1}, \ldots, B_{t_n})$ is given by*

$$\frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right), \tag{4}$$

where $t_0 = 0$ and $x_0 = 0$.

Proof. We prove the theorem for $n = 2$. The general case can be proved by induction.

Let A_1, A_2 be two Borel subsets of $\mathbb{R}$, then we have

$$\begin{aligned}
\mathbb{P}(B_{t_1} \in A_1, B_{t_2} \in A_2) &= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \mathbb{P}(B_{t_2} \in A_2 | B_{t_1} = x_1) \mathrm{d}x_1 \\
&= \int_{A_1} \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{x_1^2}{2t_1}\right) \left(\int_{A_2} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left(\frac{-(x_2 - x_1)^2}{2(t_2 - t_1)}\right) \mathrm{d}x_2\right) \mathrm{d}x_1
\end{aligned}$$

where to obtain the last equality we use Theorem 7. □

Corollary 9. *Let B_t be a Markov process with transition kernel (3), $B_0 = 0$ and such that for almost every $\omega \in \Omega$ the function $t \longmapsto B_t(\omega)$ is in $C^0(\mathbb{R}_+, \mathbb{R})$, then B_t is a Brownian motion.*

Proof. We have only to prove that B_t satisfies the second property of Definition 1. Using the same reasoning of Corollary 8, we obtain that, if B_t is a Markov process with transition kernel (3), then it has finite dimensional marginals given by (4). This implies that for any $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \in \mathbb{R}_+$ we have that $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent random variables and $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$. □

1.3 The martingale problem for Brownian motion

Definition 10. *A martingale $(M_t)_{t \geqslant 0}$ wrt. a filtration $(\mathscr{G}_t)_{t \geqslant 0}$ is a process such that*

a) $\mathbb{E}|M_t| < \infty$ *for all* $t \geqslant 0$,

b) M_t *is* $\mathscr{G}_t$ *measurable for all* $t \geqslant 0$,

c) $\mathbb{E}[M_t | \mathscr{G}_s] = M_s$ *for all* $0 \leqslant s \leqslant t$.

Proposition 11. *$(B_t)_{t \geqslant 0}$ is a continuous martingale.*

Proof. Easy. □

Next we describe a caracterisation of the Brownian motion as solution of a *martingale problem.*

Theorem 12. *Let $(X_t)_{t \geqslant 0}$ be a continuous process such that $X_0 = 0$ and for each $f \in C_b^2(\mathbb{R}; \mathbb{R})$ define*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \frac{1}{2} f''(X_s) \mathrm{d}s.$$

Then M_t^f is a martingale wrt. the filtration $\mathscr{F}_t = \sigma(X_s : s \in [0, t])$ for all $f \in C_b^2(\mathbb{R}; \mathbb{R})$ iff $(X_t)_{t \geqslant 0}$ is is a standard Brownian motion.

Proof. Let us prove the direct implication. Take the function $f(x) = \exp(i\lambda x)$ (split it in real and imaginary part to apply the assumptions), then we have that

$$M_t^\lambda = e^{i\lambda X_t} - e^{i\lambda X_0} + \frac{\lambda^2}{2} \int_0^t e^{i\lambda X_s} \mathrm{d}s, \qquad t \geqslant 0,$$

is a martingale, from this we deduce that $\mathbb{E}[M_t^\lambda] = \mathbb{E}[M_0^\lambda] = 0$ since $M_0^\lambda = 0$. Taking expectation we have

$$\mathbb{E} e^{i\lambda X_t} - \mathbb{E} e^{i\lambda X_0} + \frac{\lambda^2}{2} \int_0^t \mathbb{E} e^{i\lambda X_s} \mathrm{d}s = 0$$

and solving this simple differential equation for $\varphi(t) = \mathbb{E} e^{i\lambda X_t}$ we obtain that $\varphi(t) = e^{-\lambda^2/2}$, meaning that $X_t \sim \mathscr{N}(0, t)$. Now consider

$$e^{-i\lambda X_s}(M_t^\lambda - M_s^\lambda) = e^{i\lambda(X_t - X_s)} - 1 + \frac{\lambda^2}{2} \int_s^t e^{i\lambda(X_r - X_s)} \mathrm{d}r$$

and observe that by the martingale property of $(M_t^\lambda)_t$ we have, for all $\mu \in \mathbb{R}$,

$$\mathbb{E}[e^{i\mu X_s} e^{-i\lambda X_s}(M_t^\lambda - M_s^\lambda)] = \mathbb{E}[e^{i\mu X_s} e^{-i\lambda X_s} \mathbb{E}[(M_t^\lambda - M_s^\lambda) | \mathscr{F}_s]] = 0$$

so we conclude that the funciton $\varphi(t,s)=\mathbb{E}[e^{i\mu X_s}e^{i\lambda(X_t-X_s)}]$ satisfies the integral equation

$$\varphi(t,s)-\mathbb{E}[e^{i\mu X_s}]+\frac{\lambda^2}{2}\int_s^t \varphi(r,s)\mathrm{d}r=0$$

from which is again easy to deduce that $\varphi(t,s)=\mathbb{E}[e^{i\mu X_s}]e^{-\frac{\lambda^2}{2}(t-s)}=e^{-\frac{\mu^2}{2}s-\frac{\lambda^2}{2}(t-s)}$. So the random variables (X_s,X_t-X_s) are independent and with centred Gaussian law with variances s and $t-s$ respectively. Following a similar argument via an induction procedure one is able to establish tha for all $0=t_0\leqslant t_1\leqslant\cdots\leqslant t_n$ the r.v. $(X_{t_k}-X_{t_{k-1}})_{k=1,\ldots,n}$ are independent and with Gaussian law of mean zero and variance t_k-t_{k-1}. As a conserquence, being X continuous and $X_0=0$ we can deduce that X is a standard Brownian motion.

In order to prove the converse we need to start from

$$M_t^f:=f(X_t)-f(X_0)-\int_0^t\frac{1}{2}f''(X_s)\mathrm{d}s.$$

It is clear that this is an integrable random variable and that it is $\mathscr{F}_t$ measurable for all $t\geqslant 0$. So Let us check the martingale property, fix $s\leqslant t$ and consider

$$\begin{aligned}\mathbb{E}[M_t^f-M_s^f|\mathscr{F}_s]&=\mathbb{E}\left[f(X_t)-f(X_s)-\int_s^t\frac{1}{2}f''(X_r)\mathrm{d}r\middle|\mathscr{F}_s\right]\\&=\mathbb{E}[f(X_t)|\mathscr{F}_s]-f(X_s)-\int_s^t\frac{1}{2}\mathbb{E}[f''(X_r)|\mathscr{F}_s]\mathrm{d}s\\&=(P_{t-s}f)(X_s)-f(X_s)-\int_s^t\frac{1}{2}P_{r-s}f''(X_s)\mathrm{d}s\end{aligned}$$

where we used the fact that $(X_t)_t$ is a Brownian motion and therefore a Markov process with a transition kernel P_t. In order to conclude that this quantity is 0 a.s. it is enough to prove that

$$(P_{t-s}f)(x)-f(x)-\int_s^t\frac{1}{2}P_{r-s}f''(x)\mathrm{d}s=0$$

for all $x\in\mathbb{R}$. Using the explicit form of P_t given in eq. (3) it is not difficult to check that

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_{t-s}f)(x)=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(P_{t-s}f)(x)=\frac{1}{2}(P_{t-s}f'')(x)$$

where also we used an integration by parts. Therefore the claim follows since $P_tf(x)\to f(x)$ as $t\to 0$. □

2 Lévy construction of Brownian motion

2.1 Haar and Schauder functions

We define *Haar* functions $h_k^n(t)$ for $n=0,1,\ldots\in\mathbb{N}$ and $k=0,\ldots,2^{n-1}-1$ in the following way: for $n=0$ we put $h_0^0(t)=1$ and for $n\neq 0$ we write

$$h_k^n(t):=2^{\frac{n-1}{2}}\left(\mathbb{I}_{\left[\frac{2k}{2^n},\frac{2k+1}{2^n}\right)}(t)-\mathbb{I}_{\left[\frac{2k+1}{2^n},\frac{2k+2}{2^n}\right)}(t)\right).$$

We define also *Schauder* functions as

$$e_n^k(t) := \int_0^t h_n^s(s)\mathrm{d}s.$$

Lemma 13. *The set of Haar functions forms an orthonormal basis of* $L^2([0,1])$.

Proof. The orthonormality is a consequence of the fact that $h_k^n(t)$ and $h_{k'}^n(t)$ are supported in different sets when $k \neq k'$, and that $h_n^k(t)$ has integral 0 on the dyadic set of the form $\left[\frac{k'}{2^{n-1}}, \frac{k'+1}{2^{n-1}}\right]$ (for any $k' \in \mathbb{N}$).

In order to prove that the Haar functions form a complete basis of $L^2([0,1])$ we have only to prove that for any function $f \in L^2([0,1])$ such that $\int_0^1 f(t)h_k^n(t) = 0$ for all n, k we have $f = 0$.

Consider the probability space $([0,1], \mathscr{B}, \mathrm{d}x)$ (where $\mathscr{B}$ is the complete σ-algebra generated by Borel sets and $\mathrm{d}x$ is the Lebesgue measure) and consider the filtration $\mathscr{B}_n = \left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], k = 0, \ldots, 2^n - 1\right\}$, with $n \in \mathbb{N}$. It is easy to see that $\mathscr{B}_\infty := \sigma(\mathscr{B}_n, n \in \mathbb{N}) = \mathscr{B}$.

One can prove by induction that the functions of the form $\mathbb{I}_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}$ are in the span of $V_n = \{h_\ell^m\colon 0 \leqslant m \leqslant n, \ell = 0, \ldots, 2^{m-1} - 1\}$. Therefore if $\int_0^1 f(t)h_k^n(t)\mathrm{d}t = 0$ for $n \leqslant N$ then $\int_0^1 f(t)\mathbb{I}_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}\mathrm{d}t = 0$ for $n \leqslant N$. As a consequence $\int_0^1 f(t)g(t)\mathrm{d}t = 0$ for all $g\colon [0,1] \to \mathbb{R}$ bounded and $\mathscr{G}_n$ measurable and for all $n \geqslant 0$. This implies that

$$f_n = \mathbb{E}[f|\mathscr{B}_n] = 0, \qquad n \geqslant 0.$$

The family of random variable $(f_n)_{n \geqslant 0}$ is by construction a discrete martingale and we have also $\mathbb{E}[f_n^2] = 0$. So this martingale is bounded in $L^2([0,1])$ and by the L^2 convergence theorem for martingales we have that $f_n \to f_\infty$ in L^2 and a.s. and that moreover $f_\infty = \mathbb{E}[f|\mathscr{B}_\infty] = \mathbb{E}[f|\mathscr{B}] = f$. Therefore since $f_n = 0$ for all n we deduce that $f = 0$. □

Lemma 14. *We have that* $\sup_{t \in [0,1]} |e_n^k(t)| \leqslant 2^{-\frac{n-1}{2}}$ *and the series*

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{2^{n-1}-1} e_n^k(t) e_n^k(s) \right) = \min(t,s) \tag{5}$$

is absolutely convergent and it is equal to $\min(t,s)$.

Proof. The bound on $|e_n^k(t)|$ follows by a direct computation. In order to prove equality (5) we note that $\int_0^1 \mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau = e_n^k(t)$ (and a similar relation holds for $e_n^k(s)$). Using Parseval identity for orthonormal bases in an Hilbert space we obtain

$$\begin{aligned}
\min(t,s) &= \int_0^1 \mathbb{I}_{[0,t]}(\tau)\mathbb{I}_{[0,s]}(\tau)\mathrm{d}\tau \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2^{n-1}-1} \int_0^1 \mathbb{I}_{[0,t]}(\tau)h_n^k(\tau)\mathrm{d}\tau \int_0^1 \mathbb{I}_{[0,s]}(\tau)h_n^k(\tau)\mathrm{d}\tau \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2^{n-1}-1} e_n^k(t)e_n^k(s) \right)
\end{aligned}$$

and the previous series is absolutely convergent. □

2.2 Lévy construction of Brownian motion

We will prove that we can define a process $(B_t)_{t\in[0,1]}$ satisfying all the requirement of Definition 1 provided we restrict them to the interval $[0,1]$. This will be a Brownian motion on $[0,1]$.

Once we have a BM on $[0,1]$ we can then easily construct a sequence of independent Brownian motions $\tilde{B}_t^1,\dots,\tilde{B}_t^n$ defined on $[0,1]$ and define the process $(B_t)_{t\geqslant 0}$ as

$$B_t = \sum_{k=1}^{n-1} \tilde{B}_1^k + \tilde{B}_{t-n+1}^n, \qquad \text{if } t\in[n-1,n)$$

and easily check that this is a Brownian motion on $[0,1]$.

Alternatively, think to a way to start from a single BM $(\tilde{B}_t)_{t\in[0,1]}$ on $[0,1]$ and produce (by cutting, scaling and gluing) a BM on $\mathbb{R}_+\dots$

From now on we will concentrate on constructing a BM on $[0, 1]$. Let $(Z_{n,k})_{n,k}$ be a sequence of independent random variables such that $Z_{n,k}\sim\mathcal{N}(0,1)$. Consider the following sequence of stochastic processes

$$B_t^N(\omega) = \sum_{n=0}^{N}\left(\sum_{k=0}^{2^{n-1}-1} Z_{n,k}(\omega)e_k^n(t)\right).$$

By construction, for all $\omega\in\Omega$ the function $t\in[0,1]\mapsto B_t^N(\omega)$ is continuous and $B_0^N(\omega)=0$.

Theorem 15. *The sequence of random variables $(B^N)_N$ is almost surely convergent in $C([0,1];\mathbb{R})$. The limit B of $(B^N)_N$ is a Brownian motion on $[0,1]$.*

Proof. First we prove that the sequence of functions $t\mapsto B_t^N(\omega)$ is uniformly convergent in $C([0,1],\mathbb{R})$ for almost every $\omega\in\Omega$.

First of all we need a uniform control of the family $(Z_{n,k})_{n,k}$. Fix $0<\lambda<1/2$, let

$$Q(\omega) := \sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}-1} 2^{-2n} e^{\lambda|Z_{n,k}(\omega)|^2}$$

and note that $\mathbb{E}e^{\lambda|Z_{n,k}|^2}=C<\infty$ uniformly in n,k and by Fubini,

$$\mathbb{E}[Q] = \sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}-1} 2^{-2n}\mathbb{E}e^{\lambda|Z_{n,k}(\omega)|^2} = C\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}-1} 2^{-2n} < \infty.$$

Therefore the set $A=\{\omega\in\Omega: Q(\omega)<\infty\}$ has probability 1, moreover there exists a finite constant C (different from the previous) such that

$$\sup_{k=0,\dots,2^{n-1}-1} |Z_{n,k}(\omega)| \leqslant C(n^{1/2}+\log^{1/2}Q(\omega)), \qquad n\geqslant 0.$$

This is the uniform control we were looking for. Now we observe that for $N \geqslant M$ we have

$$|B_t^N(\omega) - B_t^M(\omega)| = \left| \sum_{n=M+1}^{N} \sum_{k=0}^{2^{n-1}-1} Z_{n,k}(\omega) e_k^n(t) \right| \leqslant \sum_{n=M+1}^{N} \left(\sup_{k=0,\ldots,2^{n-1}-1} |Z_{n,k}(\omega)| \right) \sum_{k=0}^{2^{n-1}-1} |e_k^n(t)|$$

$$\leqslant \sum_{n=M+1}^{N} C(n^{1/2} + \log^{1/2} Q(\omega)) \sum_{k=0}^{2^{n-1}-1} |e_k^n(t)|$$

and that

$$\sup_{t\in[0,1]} \sum_{k=0}^{2^{n-1}-1} |e_n^k(t)| = \sup_{t\in[0,1]} \sup_k |e_k^n(t)| \leqslant 2^{-\frac{n-1}{2}}$$

since the functions $(e_k^n)_k$ for fixed n have disjoint support. Finally we have

$$\sup_{t\in[0,1]} |B_t^N(\omega) - B_t^M(\omega)| = C \sum_{n=M+1}^{N} (n^{1/2} + \log^{1/2} Q(\omega)) 2^{-\frac{n-1}{2}} \leqslant C'(1 + \log^{1/2} Q(\omega))$$

and also that

$$\lim_{M\to\infty} \sup_{N \geqslant M} \sup_{t\in[0,1]} |B_t^N(\omega) - B_t^M(\omega)| = 0$$

whenever $Q(\omega) < \infty$. We conclude that $(B^N(\omega))$ is a Cauchy sequence on $C([0,1], \mathbb{R})$ for all $\omega \in A$ and we let $B_t(\omega) = \lim_N B_t^N(\omega)$ if $\omega \in A$ and $B_t(\omega) = 0$ otherwise. Then we have $B^N \to B$ in $C([0,1]; \mathbb{R})$ almost surely since $\mathbb{P}(A) = 1$.

We have that B_t satisfies the condition 1 and 3 of Definition 1. In order to prove that B_t satisfies property 2 of Definition 1 we prove that B_t is a Gaussian process such that $\mathbb{E}[B_t] = 0$ and $\mathrm{cov}(B_t, B_s) = \min(s,t)$. Using Corollary 6, this is equivalent to prove that B_t is a Brownian motion.

Note that $(B_t^N)_{t\in[0,1]}$ is a Gaussian process since it is a linear combination of independent Gaussian random variables (prove it). We have $\mathbb{E}[B_t^N] = 0$ and

$$\mathbb{E}[B_t^N B_s^N] = \sum_{n=0}^{N} \sum_{k=0}^{2^{n-1}-1} e_k^n(t) e_k^n(s) \to \min(t,s) = t \wedge s$$

as $N \to \infty$.

Fix $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ then by dominated convergence we have

$$\mathbb{E}[e^{i\sum \lambda_k B_{t_k}}] = \lim_N \mathbb{E}[e^{i\sum \lambda_k B_{t_k}^N}] = \lim_N \exp\left(i\mathbb{E}\left[\sum \lambda_k B_{t_k}^N\right] - \frac{1}{2}\mathrm{Var}\left(\sum \lambda_k B_{t_k}^N\right) \right)$$

moreover $\mathbb{E}[\sum \lambda_k B_{t_k}^N] = 0$ and

$$\mathrm{Var}\left(\sum_k \lambda_k B_{t_k}^N \right) = \sum_{k,\ell} \lambda_k \lambda_\lambda \mathbb{E}(B_{t_k}^N B_{t_\ell}^N) = \sum_{k,\ell} \lambda_k \lambda_\lambda \mathbb{E}(B_{t_k}^N B_{t_\ell}^N) \to \sum_{k,\ell} \lambda_k \lambda_\lambda (t_k \wedge t_\ell)$$

pointwise as $N \to \infty$. We can conclude that

$$\mathbb{E}[e^{i\sum \lambda_k B_{t_k}}] = \exp\left(-\frac{1}{2}\sum_{k,\ell} \lambda_k \lambda_\lambda (t_k \wedge t_\ell)\right)$$

and therefore that $(B_t)_{t\in[0,1]}$ is a Gaussian process with mean zero and covariance function $t \wedge s$, that is a BM. □