V3F1 Elements of Stochastic Analysis – Problem Sheet 0

This sheet will be solved during the tutorials of the week of October 14th. It will not be evaluated.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space. Recall that the conditional expectation $\mathbb{E}[X|\mathscr{G}]$ of an integrable random variable X wrt. the σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ is any random variable Z which is integrable and $\mathscr{G}$-measurable and for which

$$\mathbb{E}[ZY] = \mathbb{E}[XY]$$

for all $Y \in L^1(\mathbb{P}, \mathscr{G})$. It is a.s. unique.

Exercise 1. For all $X, Y \in L^1(\mathscr{F})$ and all sub-σ-algebras $\mathscr{G}, \mathscr{H} \subseteq \mathscr{F}$ prove the following properties of the conditional expectation.

1. Linearity: $\mathbb{E}[\lambda X + \mu Y|\mathscr{G}] = \lambda \mathbb{E}[X|\mathscr{G}] + \mu \mathbb{E}[Y|\mathscr{G}]$ for all $\lambda, \mu \in \mathbb{R}$;
2. Positivity: $X \geqslant 0 a.s. \Rightarrow \mathbb{E}[X|\mathscr{G}] \geqslant 0 a.s.$;
3. Monotone convergence: $0 \leqslant X_n \nearrow X a.s. \Rightarrow \mathbb{E}[X_n|\mathscr{G}] \nearrow \mathbb{E}[X|\mathscr{G}] a.s.$;
4. Jensen's inequality: for all convex $\varphi: \mathbb{R} \to \mathbb{R}$: $\mathbb{E}[\varphi(X)|\mathscr{G}] \geqslant \varphi(\mathbb{E}[X|\mathscr{G}])$;
5. Contractivity in L^p: $\|\mathbb{E}[X|\mathscr{G}]\|_p \leqslant \|X\|_p$ for all $p \in [1, \infty]$,
6. Telescoping: If $\mathscr{H}$ is a sub-σ-algebra of $\mathscr{G}$ then
$$\mathbb{E}[\mathbb{E}[X|\mathscr{G}]|\mathscr{H}] = \mathbb{E}[X|\mathscr{H}] = \mathbb{E}[\mathbb{E}[X|\mathscr{H}]|\mathscr{G}];$$
7. If $Z \hat{\in} \mathscr{G}$, $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|XZ|] < +\infty$ then $\mathbb{E}[XZ|\mathscr{G}] = Z\,\mathbb{E}[X|\mathscr{G}]$.

Exercise 2. Assume there exists a partition $A_1, \ldots, A_n$ of Ω (i.e. the sets are pairwise disjoint and cover all Ω) and let $\mathscr{G} = \sigma(A_1, \ldots, A_n)$. Let $X \in L^1(\mathbb{P}, \mathscr{F})$.

a) Describe $\mathscr{G}$.

b) Show that $\mathbb{E}[X|\mathscr{G}] = \sum_i \mathbb{E}[X|A_i] \mathbb{1}_{A_i}$ with $\mathbb{E}[X|A_i] := \mathbb{E}[X \mathbb{1}_{A_i}] / \mathbb{E}[\mathbb{1}_{A_i}]$.

Exercise 3. Prove the following.

a) If X is independent of the σ-algebra $\mathscr{G}$, then $\mathbb{E}[X|\mathscr{G}]$ is almost surely constant and

$$\mathbb{E}[X|\mathscr{G}] = \mathbb{E}(X), \qquad \text{a.s.}$$

b) If $\mathscr{H}$ and $\mathscr{G}$ are independent, X is $\mathscr{G}$-mesurable and $\mathscr{G}' \subseteq \mathscr{G}$, then

$$\mathbb{E}[X|\mathscr{H}, \mathscr{G}'] = \mathbb{E}[X|\mathscr{G}'].$$

c) If $X_1, \ldots, X_n$ is a family of independent r.v. and $f(X_1, \ldots, X_n) \in L^1$ then

$$\mathbb{E}[f(X_1, \ldots, X_n)|X_1] = \varphi(X_1)$$

where $\varphi(x) := \mathbb{E}[f(x, X_2, \ldots, X_n)]$.