V3F1 Elements of Stochastic Analysis – Problem Sheet 10

Distributed December 19th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday January 9th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 2] Prove that if we have adapted, continuous f, g, f', g' such that

$$f_t \mathrm{d}t + g_t \mathrm{d}B_t = f_t' \mathrm{d}t + g_t' \mathrm{d}B_t, \qquad t \geqslant 0,$$

then we must have $f = f'$ and $g = g'$ almost surely. (Here B is a Brownian motion).

Exercise 2. [Pts 2+2+2+2] Let $(B_t)_{t \geqslant 0}$ be a 3-dimensional Brownian motion started at $x \neq 0$. Show that

a) $X_t = |B_t|^{-1}$ is a local martingale;

b) $(X_t)_{t \geqslant 0}$ is uniformly integrable. (Hint: study $\mathbb{E}[X_t^p]$)

c) $(X_t)_{t \geqslant 0}$ is not a martingale.

d) $B_t \to \infty$ almost surely.

Exercise 3. [Pts 2+2+2] Let $(B_t)_{t \geqslant 0}$ be a one dimensional Brownian motion. Find the SDEs satisfied by the following processes: (for all $t \geqslant 0$)

a) $X_t = B_t / (1 + t)$,

b) $X_t = \sin(B_t)$

c) $(X_t, Y_t) = (a \cos(B_t), b \sin(B_t))$ where $a, b \in \mathbb{R}$ with $ab \neq 0$

Exercise 4. [Pts 4] Give an explicit solution of the following one–dimensional SDE

$$\mathrm{d}X_t = \frac{1}{2} X_t \mathrm{d}t + \sqrt{1 + X_t^2} \mathrm{d}B_t, \qquad X_0 = x \in \mathbb{R}$$

where $(B_t)_{t \geqslant 0}$ is a standard Brownian motion.

We wish you a good start into the new year!