# V3F1 Elements of Stochastic Analysis – Problem Sheet 11

Distributed January 9th, 2020. In groups of 2. Solutions have to be handed in before 4pm on Thursday January 16th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework. (revised version 10/01/2020)

**Exercise 1.** [Pts 2+3+2] Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Define the process

$$X_t = a(t)\left(x_0 + \int_0^t b(s)\mathrm{d}B_s\right)$$

where $a, b\colon \mathbb{R}_+ \to \mathbb{R}$ are differentiable functions with $a(0) = 1$ and $a(t) > 0$. Compute the SDE satisfied by this process.

b) Use (a) to find an explicit solution for the SDEs in eqns.~(1),(2),(3):

$$\begin{cases} \mathrm{d}X_t &= -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t \qquad t \in [0, T] \\ X_0 &= x_0 \end{cases} \tag{1}$$

where $\alpha, \sigma, T$ are positive constants.

$$\begin{cases} \mathrm{d}X_t &= -\frac{X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t \qquad t \in [0, 1) \\ X_0 &= 0 \end{cases} \tag{2}$$

$$\begin{cases} \mathrm{d}X_t &= tX_t \mathrm{d}t + e^{t^2/2}\mathrm{d}B_t \qquad t \in [0, T] \\ X_0 &= 1 \end{cases} \tag{3}$$

c) Are the solutions of the SDEs in (b) strong and pathwise unique?

**Exercise 2.** [Pts 2+2+2] Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion.

a) Given $f \in C(\mathbb{R}_+)$, prove that $X_t = \int_0^t f(s)\mathrm{d}B_s$ is a Gaussian random variable with mean 0 and variance $\int_0^t f(u)^2 \mathrm{d}u$ for all $t \geqslant 0$.

b) The Ornstein–Uhlenbeck process $(X_t)_{t\geqslant 0}$ is defined as the solution to the SDE

$$\begin{cases} \mathrm{d}X_t &= (-\alpha X_t + \beta)\mathrm{d}t + \sigma \mathrm{d}B_t \qquad t \geqslant 0 \\ X_0 &= x_0 \end{cases} \tag{4}$$

where $\alpha, \sigma$ are positive constant and $\beta, x_0 \in \mathbb{R}$. Find the explicit solution to the SDE (4).

c) Prove that $X_t$ converges in distribution as $t \to \infty$ to a Gaussian random variable with mean $\beta / \alpha$ and variance $\sigma^2 / 2\alpha$.

**Exercise 3.** [Pts 3+2+2] Let $(B_t)_{t\geqslant 0}$ be a 2-dimensional Brownian motion and $X$ a two-dimensional stochastic process solution to the SDE

$$\begin{cases} \mathrm{d}X_t &= AX_t \mathrm{d}t + \mathrm{d}B_t \qquad t \geqslant 0 \\ X_0 &= \xi \end{cases} \tag{5}$$

where $\xi$ is a random variable in $\mathbb{R}^2$ independent of $B$ and

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

with $\alpha \in \mathbb{R}$.

a) Let $\phi(t)$ be a $2 \times 2$ matrix that satisfies the ODE

$$\dot{\phi}(t) = A\phi(t), \qquad \phi(0) = \mathbb{I}_2$$

where $\mathbb{I}_2$ is the $2 \times 2$ identity matrix. Show that $\phi(t) = e^{At} = \sum_{n \geqslant 0} A^n \frac{t^n}{n!}$ and calculate $\phi(t)$ explicitly. Find $\phi(t)^{-1}$ (inverse matrix).

b) Verify that

$$X_t = \phi(t)\left(\xi + \int_0^t \phi(s)^{-1} \mathrm{d}B_s\right)$$

solves the SDE (5).

c) Calculate the explicit solution of (5).