V3F1 Elements of Stochastic Analysis – Problem Sheet 12

Distributed January 17th, 2020. In groups of 2. Solutions have to be handed in before 4pm on Thursday January 24th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework. (revised version 18/1/2020)

Exercise 1. [Pts 3+3+3]

a) Show that there is a unique strong solution X_t of the 1-dimensional SDE

$$\mathrm{d}X_t = \log(1+X_t^2)\mathrm{d}t + \mathbb{1}_{\{X_t>0\}}X_t\,\mathrm{d}B_t, \qquad X_0 = x.$$

b) Let b be a function in $C^1(\mathbb{R},\mathbb{R})$ such that $b(y) \geqslant \delta > 0$. Solve the 1-dimensional SDE

$$\mathrm{d}X_t = \frac{1}{2}b(X_t)\,b'(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}B_t, \qquad X_0 = x.$$

(*Hint:* define $h(y) = \int_0^y \frac{\mathrm{d}z}{b(z)}$ and consider the process $Y = h(X)$.)

c) Let $b\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a measurable function satisfying $|b(t,x)| \leqslant C(t)$, where $C \in L^2([0,T])$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ a Lipshitz function such that for all $x \in \mathbb{R}^d$ the matrix $\sigma(x)$ is invertible and its inverse satisfies $\|\sigma^{-1}(x)\| \leqslant \lambda$ for some $\lambda < \infty$. Show uniqueness in law of weak solutions to the SDE

$$\mathrm{d}X_t = b(t,X_t)\,\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t\in[0,T]$$

(*Hint:* use Girsanov's transformation similarly to the case $\sigma(x) = \mathbb{1}_{d\times d}$)

Exercise 2. [Pts 3+3+4] Consider the one dimensional SDE

$$\mathrm{d}X_t = -X_t^3\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x$$

where B is a standard Brownian motion.

a) Let $f(t,x) = (1+|x|^2)$ and $T_L = \inf\{t \geqslant 0 : |X_t| > L\}$. Use Ito formula to show that there exists a constant λ such that the process $Z_t := e^{-\lambda(t\wedge T_L)}f(X_{t\wedge T_L})$ is a supermartingale.

b) Deduce that $\mathbb{P}(T_L \leqslant t) \to 0$ as $L \to \infty$.

c) Conclude that solutions of the SDE cannot explode (that is $\zeta := \sup_L T_L = \infty$ a.s.).

Exercise 3. [Pts 4] **(Population growth in a stochastic, crowded environment)** The nonlinear SDE

$$\mathrm{d}X_t = rX_t(K - X_t)\,\mathrm{d}t + \beta X_t\,\mathrm{d}B_t, \qquad X_0 = x > 0, \tag{1}$$

is often used as a model for the growth of a population of size X_t in a stochastic, crowded environment. The constant $K > 0$ is called the *carrying capacity* of the environment, the constant $r \in \mathbb{R}$ is a measure of the quality of the environment and the constant $\beta \in \mathbb{R}$ is a measure of the size of the noise in the system. Verify that

$$X_t = \exp\left\{\left(rK - \frac{1}{2}\beta^2\right)t + \beta B_t\right\}\left(\frac{1}{x} + r\int_0^t \exp\left\{\left(rK - \frac{1}{2}\beta^2\right)s + \beta B_s\right\}\mathrm{d}s\right)^{-1}, \quad t \geqslant 0,$$

is the unique (strong) solution to (1).