V3F1 Elements of Stochastic Analysis – Problem Sheet 2

Distributed October 17th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday October 24th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [3+2 Pts] Let $(X_t)_{t\geqslant 0}$ be a continuous Gaussian process with independent increments and such that $\mathrm{Var}(X_{t+h}-X_t)=\mathrm{Var}(X_h)$ for all $t,h\geqslant 0$ and $\mathrm{Var}(X_0)=0$. We want to prove that $(X_t)_{t\geqslant 0}$ it is a Brownian motion. Let $\varphi(t):=\mathrm{Var}(X_t)$.

a) Prove that φ is a continuous function.

b) Prove that φ solves $\varphi(t+s)=\varphi(t)+\varphi(s)$ and conclude that it is a linear function.

Therefore $(X_t)_{t\geqslant 0}$ has to be a Brownian motion.

Exercise 2. [3 Pts] Let $(B_t)_{t\geqslant 0}$ be a Brownian motion, $(\mathscr{F}_t)_{t\geqslant 0}$ its canonical filtration (i.e. $\mathscr{F}_t=\sigma(B_s, s\in[0,t])$) and $f:\mathbb{R}\to\mathbb{R}$ a bounded and measurable function. Prove that

$$\mathbb{E}\left[\int_s^t f(B_r)\mathrm{d}r\middle|\mathscr{F}_s\right]=\int_s^t \mathbb{E}[f(B_r)|\mathscr{F}_s]\mathrm{d}r.$$

(Hint: show that $(r,\omega)\mapsto\mathbb{E}[f(B_r)|\mathscr{F}_s](\omega)$ is jointly measurable and bounded and use the definition of conditional expectation)

Exercise 3. [2 Pts] Let $(B_t)_{t\geqslant 0}$ be a Brownian motion, fix $T>0$ and consider the process

$$X_t=B_t\mathbb{1}_{t\leqslant T}+[B_T+Z(B_t-B_T)]\mathbb{1}_{t>T}$$

where Z is a random variable independent of B and with distribution $\mathbb{P}(Z=\pm 1)=1/2$. Show that $(X_t)_t$ is still a Brownian motion.

Exercise 4. [Pts 1+3+2+2+2] Let $(B_t)_{t\geqslant 0}$ be a Brownian motion. We want to prove that for every $x\in\mathbb{R}$ there exists a measure μ^x on $C([0,1];\mathbb{R})$ such that for all measurable and bounded $f:C([0,1];\mathbb{R})\to\mathbb{R}$ we have

$$\mathbb{E}[f((B)_{t\in[0,1]})|B_1](\omega)=\int_{C([0,1];\mathbb{R})} f(\eta)\,\mu^{B_1(\omega)}(\mathrm{d}\eta).$$

The measure μ^x is called the Brownian bridge.

a) Show that the process $Z_t=B_t-tB_1$ is Gaussian.

b) Show that $(Z_t)_{t\geqslant 0}$ is independent from B_1.

c) Show that

$$\mathbb{E}[f((B_t)_{t\in[0,1]})|B_1]=h(B_1)$$

where $h(x)=\mathbb{E}[f((tx+Z_t)_{t\in[0,1]})]$.

d) Show that μ^x is the law of a continuous Gaussian process.

e) Give the mean and covariance of such a process.