V3F1 Elements of Stochastic Analysis – Problem Sheet 3

Distributed October 24th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday October 31st into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 1+1+2] Let G a geometric random variable such that $\mathbb{P}(G=k)=p^k(1-p)$ for all $k\geqslant 0$. Let $\mathscr{F}_n=\sigma(G\wedge(n+1))$. Show that

a) $(\mathscr{F}_n)_{n\geqslant 0}$ is a filtration;

b) $(M_n=\mathbb{1}_{G\leqslant n}-(1-p)(G\wedge n))_{n\geqslant 0}$ is a martingale for the filtration $(\mathscr{F}_n)_{n\geqslant 0}$;

c) $(Y_n=M_n^2-p(1-p)(G\wedge n))_{n\geqslant 0}$ is another martingale for the same filtration.

Exercise 2. [Pts 1+2+2+1] Let $(Y_n)_n$ and i.i.d. sequence with $Y_n\geqslant 0$ and $\mathbb{E}[Y_n]=1$. Let $X_n=\prod_{k=1}^n Y_k$ for all $n\geqslant 1$ and $X_0=1$.

a) Show that $(X_n)_n$ is a martingale wrt. the natural filtration of $(Y_n)_n$.

b) Assume that $Y_n\geqslant\delta$ for some $\delta>0$. Show that $\mathbb{E}[\log Y_1]<\infty$ and use the law of large numbers to show that if $\mathbb{P}(Y_1=1)<1$ then $X_n\to 0$ almost surely.

c) Let $Z_n=\max(\delta, Y_n)$. Show that there exists $\delta>0$ such that $\mathbb{E}[\log Z_n]<\infty$ and conclude that, if $\mathbb{P}(Y_1=1)<1$ then $X_n\to 0$ almost surely, without additional hypothesis on Y.

d) Conclude that, in general, the convergence $X_n\to X_\infty$ in Doob's martingale convergence theorem is not in L^1 but only almost surely.

Exercise 3. [Pts 2+2] Let $(X_n)_{n\geqslant 0}$ be a martingale. Show that these two statements are equivalent:

a) There exists positive martingales $(X_n^+)_{n\geqslant 0}$ and $(X_n^-)_{n\geqslant 0}$ such that $X_n=X_n^+-X_n^-$;

b) X is bounded in L^1.

(Hint: consider $\lim_n\mathbb{E}[X_{m+n}^+|\mathscr{F}_m]$ for $m\geqslant 0$)

Exercise 4. [Pts 3+3] (Robbins–Monroe algorithm) Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence with repartition function $F(t)=\mathbb{P}(X_1\leqslant t)$ and let $(\mathscr{F}_n)_{n\geqslant 0}$ the natural filtration of X with $\mathscr{F}_0=\{\emptyset,\Omega\}$. We will assume that F is continuous and for all $\alpha\in(0,1)$ we let q_α the unique solution to $F(q_\alpha)=\alpha$ (the α-th quantile of F). Let $(Y_n)_{n\geqslant 1}$ the sequence defined by induction via

$$Y_{n+1}=Y_n-\gamma_n(\mathbb{1}_{X_{n+1}\leqslant Y_n}-\alpha),\qquad n\geqslant 0, \tag{1}$$

with Y_0 a fixed, arbitrary constant and $\alpha\in(0,1)$. The sequence $(\gamma_n)_{n\geqslant 0}$ is positive and decreasing and such that $\sum_n\gamma_n^2<\infty$, $\sum\gamma_n=+\infty$. The recurrence (1) is a statistical algorithm to approximate the α-th quantile q_α via observations involving only the random variables $(\mathbb{1}_{X_n\leqslant\ell_n})_n$ for a sequence of random levels $(\ell_n)_n$. It is called the Robbins-Monroe algorithm. We want to show that $Y_n\to q_\alpha$ almost surely.

a) Let $(Z_n)_n$ the sequence defined by $Z_n=(Y_n-q_\alpha)^2$. Compute $\mathbb{E}[Z_{n+1}|\mathscr{F}_n]$ and show that there exists an increasing and bounded sequence $(U_n)_{n\geqslant 1}$ such that $W_n=Z_n-U_n$ satisfy

$$0\leqslant\gamma_n(Y_n-q_\alpha)(F(Y_n)-\alpha)\leqslant W_n-\mathbb{E}[W_{n+1}|\mathscr{F}_n].$$

b) Show that $(W_n)_n$ converges almost surely and that the series

$$\sum_n\gamma_n(Y_n-q_\alpha)(F(Y_n)-\alpha)$$

converges in L^1 and almost surely, and that from this we can deduce $Y_n\to q_\alpha$ almost surely.