V3F1 Elements of Stochastic Analysis – Problem Sheet 4

Distributed October 31st, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday November 7th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 2+2]

a) Let $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space, let $\mathscr{G} \subseteq \mathscr{F}$ a sub-σ-algebra and $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a measurable function. Let X be a r.v. independent of $\mathscr{G}$ and and Y a $\mathscr{G}$-measurable r.v. Show that

$$\mathbb{E}[F(X, Y)|\mathscr{G}](\omega) = \int_\Omega F(X(\omega'), Y(\omega))\mathbb{P}(\mathrm{d}\omega').$$

(Hint: show it for appropriate F and then extend via the Monotone Class theorem)

b) Let T_1, T_2 be two independent exponential r.v. with parameter $\alpha > 0$ and let $X = T_1 \wedge T_2$. Compute $\mathbb{E}[X|T_1]$.

Exercise 2. [Pts 3+2+3] Let $\lambda, \tilde{\lambda}$ be finite measures on the measure space $(W, \mathscr{W})$ and let $\alpha: W \to [0, 1]$.

a) Show that, if $\Lambda, \tilde{\Lambda}$ are two independent Poisson point processes (PPP) with respective intensities $\lambda, \tilde{\lambda}$ then $\Lambda + \tilde{\Lambda}$ is a PPP with intensity $\lambda + \tilde{\lambda}$.

b) Let Λ a PPP with intensity λ and let $g: W \to \mathbb{R}^d$ a measurable function integrable wrt. λ. Determine the law of the random variable

$$H = \int_W g(x)\Lambda(\mathrm{d}x).$$

c) Let $N, (X_n)_n, (U_n)_n$ independent r.v.s such that $N \sim \mathrm{Poi}(\lambda(W))$, $X_k \sim \lambda(\cdot) / \lambda(W)$ and $U_k \sim U([0, 1])$ and set

$$\Lambda = \sum_{k=1}^N \mathbb{1}_{\alpha(X_k) \leqslant U_k} \delta_{X_k}.$$

Prove that Λ is a PPP and find its intensity.

Exercise 3. [Pts 4+4] Let $(N_t)_{t \geqslant 0}$ be a Poisson counting process with intensity 1, namely such that $N_t - N_s \sim \mathrm{Poi}(t - s)$ for all $t \geqslant s$. For all $f: \mathbb{N} \to \mathbb{R}$ bounded let

$$M_t^f = f(N_t) - f(0) - \int_0^t [f(N_s + 1) - f(N_s)]\mathrm{d}s.$$

a) Prove that $(M_t^f)_{t \geqslant 0}$ is a càdlàg martingale wrt. the filtration $(\mathscr{G}_t)_{t \geqslant 0}$ generated by $(N_t)_{t \geqslant 0}$. (Hint: could helpful to compute $\partial_h \mathbb{E}[f(N_{h+s})|\mathscr{G}_s]$ for $h \geqslant 0$)

b) Prove that any càdlàg process $(N_t)_t$ for which $(M_t^f)_{t \geqslant 0}$ is a càdlàg martingale for any f has to be a Poisson point process (Hint: follow the idea of proof given in the lectures in the Brownian setting).