V3F1 Elements of Stochastic Analysis – Problem Sheet 5

Distributed November 7th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday November 14th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 4] Let $(X_t)_{t\geqslant 0}$ be a positive, right-continuous, submartingale wrt. the filtration $(\mathscr{F}_t)_{t\geqslant 0}$. Show that the following statements are equivalent:

a) The family $(X_t)_{t\geqslant 0}$ is uniformly integrable;

b) The L^1-limit $\lim_{t\to\infty} X_t$ exists;

c) There exists a random variable $X_\infty \in L^1$ such that $X_t \to X_\infty$ a.s. and $(X_t)_{t\in\mathbb{R}_+\cup\{\infty\}}$ is a submartingale wrt. the filtration $(\mathscr{F}_t)_{t\in R_+\cup\{\infty\}}$ where $\mathscr{F}_\infty = \cup_{t\geqslant 0}\mathscr{F}_t$.

Exercise 2. [Pts 3+3] Consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ and $p \in (1, \infty)$. For a process $(M_t)_{t\geqslant 0}$ define $\|M\| = \sup_{t\geqslant 0} (\mathbb{E}[|M_t|^p])^{1/p}$ and consider the space

$$\mathscr{M}^p = \{(M_t)_{t\geqslant 0} \text{is a martingale}, M_0 = 0, \|M\| < \infty\}.$$

a) Show that $\mathscr{M}^p$ is a Banach space

b) Show that the map $J: \mathscr{M}^p \to L^p(\Omega, \mathscr{F}, \mathbb{P})$ given by $JM = M_\infty$ is an isometry between Banach spaces.

Exercise 3. [Pts 2+2+2] Let $(B_t)_{t\geqslant 0}$ be a d-dimensional Brownian motion (i.e. a d-dimensional vector of independent standard real-valued Brownian motions). Show that the following processes are martingales both for the canonical filtration $\mathscr{F}_t^B = \sigma(B_s: s \in [0, t])$ and the right-continuous filtration $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s^B$:

a) $(B_t^{(i)})_{t\geqslant 0}$ for $i = 1, \ldots, d$;

b) $(B_t^{(i)} B_t^{(j)} - \delta_{i,j} t)_{t\geqslant 0}$ for $i, j = 1, \ldots, d$;

c) $\left(\exp\left(\langle \lambda, B_t\rangle - \frac{1}{2}|\lambda|^2\right)\right)_{t\geqslant 0}$ for $\lambda \in \mathbb{R}^d$, where $|\lambda|$ denotes the Euclidean norm.

Exercise 4. [Pts 4] Let $t > 0$. For all $n \geqslant 0$ let $(\Pi^n)_n$ a sequence of partitions of $[0, t]$ such that $\Pi^n = \{t_k^n: k = 0, \ldots, n\}$ with $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ and $|\Pi^n| = \sup_k |t_{k+1}^n - t_k^n| \to 0$ as $n \to \infty$. For any $q \geqslant 1$ consider the q-variation process of X wrt Π^n, defined as $V_t^{(q)}(\Pi^n) \coloneqq \sum_{k=1}^n |X_{t_k} - X_{t_{k-1}}|^q$. Assume that X is a continuous, adapted process such that, for some $p > 0$ and all $t \geqslant 0$ $\lim_{n\to\infty} V_t^{(q)}(\Pi^n) = L_t$ a.s. where L_t is a positive random variable which can take the value $+\infty$.

a) Let $q \in (0, p)$ and assume that $L_t > 0$ a.s. Show that $\lim_{n\to\infty} V_t^{(p)}(\Pi^n) = +\infty$ a.s.

b) Let $q > p$ and assume that $L_t < \infty$ a.s. Show that $\lim_{n\to\infty} V_t^{(p)}(\Pi^n) = 0$ a.s.