V3F1 Elements of Stochastic Analysis – Problem Sheet 6

Distributed November 14th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday November 21st into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [2+2+2 Pts]

a) Let T, S be $(\mathscr{G}_t)_{t\geqslant 0}$ stopping times. Prove that $T\wedge S$ and $T\vee S$ are $(\mathscr{G}_t)_{t\geqslant 0}$ stopping times ($\wedge$ and $\vee$ denote resp. minimum and maximum)

b) Let T, S be $(\mathscr{G}_{t+})_{t\geqslant 0}$ stopping times with $T>0$ and $S>0$. Prove that $T+S$ is a $(\mathscr{G}_t)_{t\geqslant 0}$ stopping time.

c) If $T>0$ is a $(\mathscr{G}_t)_{t\geqslant 0}$ stopping time and S be a $(\mathscr{G}_{t+})_{t\geqslant 0}$ stopping time, prove that $T+S$ is a $(\mathscr{G}_t)_{t\geqslant 0}$ stopping time.

Exercise 2. [Pts 2+2] Let T, S be $(\mathscr{F}_t)_{t\geqslant 0}$ stopping times and let Z be an integrable random variable. Show that

a) $\mathbb{E}[Z|\mathscr{F}_T]\mathbb{1}_{T\leqslant S} = \mathbb{E}[Z|\mathscr{F}_{T\wedge S}]\mathbb{1}_{T\leqslant S}$ a.s.

b) $\mathbb{E}[\mathbb{E}[Z|\mathscr{F}_T]|\mathscr{F}_S] = \mathbb{E}[Z|\mathscr{F}_{S\wedge T}]$ a.s.

Exercise 3. [Pts 2+2+2] Let $(N_t)_{t\geqslant 0}$ be the Poisson counting process with intensity $\lambda>0$. For $a\in\mathbb{N}$ define the stopping time $T_a = \inf\{t\geqslant 0: N_t = a\}$. It can be shown easily that T_a is almost surely finite. Show that

a) $\mathbb{E}[T_a] = a/\lambda$,

b) $\mathrm{Var}[T_a] = a/\lambda^2$.

Exercise 4. [Ptr 2+2] Let $(B_t)_{t\geqslant 0}$ be a one-dimensional Brownian motion. For $a, b>0$ and $x\in\mathbb{R}$ define the stopping time $T_x = \inf\{t\geqslant 0: B_t = x\}$ and $T_{a,b} = T_{-a}\wedge T_b$.

a) Let $\theta\geqslant 0$ and $X_t^{\theta,a} = \sinh(\theta(B_t+a))e^{-\theta^2 t/2}$. Show that this is a martingale.

b) Let $\lambda\geqslant 0$. Show that

$$\mathbb{E}[e^{-\lambda T_{a,b}}] = \frac{\cosh\big((a-b)\sqrt{\lambda/2}\big)}{\cosh\big((a+b)\sqrt{\lambda/2}\big)}.$$

(Hint: consider $X_t^{\theta,a}$ and $X_t^{\theta,-b}$ and use optional sampling)

The student council of mathematics will organize the Maths Party on 28/11 in N8schicht. The presale will be held on Tue 26/11, Wed 27/11 and Thu 28/11 in the mensa Poppelsdorf. Further information can be found on fsmath.uni-bonn.de.