V3F1 Elements of Stochastic Analysis – Problem Sheet 7

Distributed November 28th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday December 5th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework. (revised 3/12/19)

Exercise 1. [Pts 2+3] Let $(B_t)_t$ a n-dimensional Brownian motion.

a) Let T be a stopping time such that $\mathbb{E}[T] < \infty$. Show that

$$\mathbb{E}[B_T] = 0, \quad \mathbb{E}[|B_T|^2] = n\mathbb{E}[T].$$

b) Let $r > 0$, $x \in \mathbb{R}^n$ and consider the stopping time $T_{x,r} := \inf\{t \geqslant 0 : |B_t - x| \geqslant r\}$. It holds that $\mathbb{E}[T_{x,r}] < \infty$. Show that

$$\mathbb{E}[T_{x,r}] = \begin{cases} \frac{r^2 - |x|^2}{n} & \text{if } |x| < r; \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. [Pts 3] Show that the following σ-algebras are the same

$$\sigma(\mathscr{E}_b) = \sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} | X \in \mathscr{E}_b \text{ and the map } (t, \omega) \mapsto X_t(\omega) \text{ is measurable}\})$$

$$\sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} | X \text{ is adapted and left-continuous on } (0, \infty)\})$$

$$\sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} | X \text{ is adapted and continuous on } [0, \infty)\})$$

Recall that $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ belongs to $\mathscr{E}_b$ iff it can be written as $X_t(\omega) = \sum_{i \geqslant 0} x_i(\omega) \mathbb{1}_{t_{i-1} < t \leqslant t_i}$ where $0 = t_0 < t_1 < t_2 < \cdots$ and $x_i : \Omega \to \mathbb{R}$ is $\mathscr{F}_{t_{i-1}}$ measurable and bounded.

Exercise 3. [Pts 2+2+2+2+2] Let M a continuous local martingale. Show that

a) If an integrable, adapted and cadlag stochastic process X satisfies $\mathbb{E}[X_0] = \mathbb{E}[X_T]$ for all bounded stopping times T, then X is a martingale.

b) If $\mathbb{E}(\sup_{s \in [0,t]} |M_s|) < \infty$ for all $t \geqslant 0$ then M is a martingale;

c) If $\mathbb{E}([M]_t) < \infty$ for all $t \geqslant 0$ then M is a martingale;

d) If M is non-negative and integrable for all $t \geqslant 0$ then M is a supermartingale;

e) If for all $t \geqslant 0$ the family $\{M_{t \wedge T} | T \text{ is a bounded stopping time}\}$ is uniformly integrable, then M is a martingale;

Exercise 4. [Pts 4] Let M be a continuous local martingale. Show that if $\mathbb{P}(\sup_{t \geqslant 0} [M]_t < \infty) = 1$ then $\lim_{t \to \infty} M_t$ exists a.s. *(Hint: show first that for a suitable sequence of stopping times $(T_n)_n$, the processes $(M_{t \wedge T_n})_{t \geqslant 0}$ converges for all n.)*