Institute for Applied Mathematics – WS19/20
Massimiliano Gubinelli

V3F1 Elements of Stochastic Analysis – Problem Sheet 8

Distributed December 5th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday December 12th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 2+2] Let B be a one-dimensional Brownian motion. Let $T \in (0, \infty)$ be a fixed time and $(\Pi_n = \{u_0, \ldots, u_n\})_n$ be a sequence of partitions of $[0, T]$ with $0 = u_0 < u_1 < \cdots < u_n = T$. Let $u_i^* = (u_{i+1} + u_i) / 2$ and $|\Pi_n| = \sup_{0 \leqslant i < n} |u_{i+1} - u_i|$.

a) Show that, in probability,

$$\lim_{|\Pi_n| \to 0} \sum_{i=0}^{n-1} |B_{u_i^*} - B_{u_i}|^2 = \frac{T}{2}.$$

b) Define the *Stratonovich integral* of B with respect of B by

$$\int_0^T B_s \circ \mathrm{d}B_s := \lim_{|\Pi_n| \to 0} \sum_{i=0}^{n-1} B_{u_i^*}(B_{u_{i+1}} - B_{u_i})$$

where the limit is undestood in probability. Show that

$$\int_0^T B_s \circ \mathrm{d}B_s = \int_0^T B_s \mathrm{d}B_s + \frac{T}{2}$$

where on the r.h.s. the is the Itô integral.

Exercise 2. [Pts 2+2+2] Let $(B_t)_{t \geqslant 0}$ be a standard Brownian motion and $(\sigma_t)_t$ and adapted process such that $\mathbb{E}[\int_0^\infty \sigma_s^2 \mathrm{d}s] < \infty$. Define

$$Z_t = \exp\left(\int_0^t \sigma_s \mathrm{d}B_s - \frac{1}{2} \int_0^t \sigma_s^2 \mathrm{d}s \right).$$

a) Use Itô formula to prove that $(Z_t)_t$ satisfies $\mathrm{d}Z_t = \sigma_t Z_t \mathrm{d}B_t$.

b) Prove that $(Z_t)_{t \geqslant 0}$ is a supermartingale.

c) If $\sigma_t = \sigma$ (constant in time), prove that $(Z_t)_{t \geqslant 0}$ is a martingale.

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Exercise 3. [Pts 3+3] Let B be a d-dimensional Brownian motion starting at $x \neq 0$. For $a > 0$ define the stopping time $T_a = \inf\{t \geqslant 0 : |B_t| = a\}$.

a) Let $d = 2$ and $0 < r < |x| < R$. Show that $(\log|B_t^{T_r \wedge T_R}|)_{t \geqslant 0}$ is a bounded martingale and prove that

$$\mathbb{P}[T_r < T_R] = \frac{\log R - \log|x|}{\log R - \log r}.$$

b) Let $d = 3$. Show that $(|B_t^{T_R \wedge T_r}|^{-1})_{t \geqslant 0}$ is a bounded martingale and that

$$\mathbb{P}[T_r < T_R] = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$

Deduce that $\mathbb{P}[T_r < \infty] = r / |x|$.

Exercise 4. [Pts 2+2]

a) Let M be a local martingale and $(f_t, g_t)_{t \geqslant 0}$ two bounded adapted processes. Let $Y_t = \int_0^t g_s \mathrm{d}M_s$ and $Z_t = \int_0^t f_s g_s \mathrm{d}M_s$. Prove that almost surely $Z_t = \int_0^t f_s \mathrm{d}Y_s$.

b) Let M be a continuous local martingale and $0 < a < b$. Prove that on a set of probability one

$$[M]_b = [M]_a \Leftrightarrow M_t = M_a \text{ for all } t \in [a, b].$$