V3F1 Elements of Stochastic Analysis – Problem Sheet 9

Distributed December 12th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday December 19th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework. (revised 7/1/20)

Exercise 1. [Pts 2+2] Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geqslant 0})$ be a standard filtered probability space. For any continuous process $X: \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and any $\lambda > 0$ let

$$\|X\|_\lambda := \sup_{T\geqslant 0} e^{-\lambda T}\left[\mathbb{E} \sup_{t\in[0,T]} |X_t|^2\right]^{1/2}.$$

Moreover let

$$\mathscr{X}_\lambda = \{X: \Omega \times \mathbb{R}_+ \to \mathbb{R}^d: t \mapsto X_t \text{ is continuous and adapted to } (\mathscr{F}_t)_t \text{ and } \|X\|_\lambda < \infty\}.$$

a) Prove that $(\mathscr{X}_\lambda, \|\cdot\|_\lambda)$ is a Banach space

b) Prove that

$$\mathbb{E} \sup_{t\geqslant 0} e^{-4\lambda t}|X_t|^2 \lesssim \|X\|_\lambda^2.$$

Exercise 2. [Pts 2+2+2+2] Let $(\Gamma_t)_{t\geqslant 0}$ be the solution to

$$\Gamma_0 = 1, \quad \mathrm{d}\Gamma_t = \Gamma_t(\beta_t \mathrm{d}t + \gamma_t \mathrm{d}B_t), \quad t \geqslant 0,$$

where $(\gamma_t)_{t\geqslant 0}, (\beta_t)_{t\geqslant 0}$ are bounded adapted processes and $(W_t)_{t\geqslant 0}$ is a Brownian motion. Assume that there is a $c > 0$ such that $\gamma_s > c$ for all $s \geqslant 0$. Fix $T \in (0, \infty)$.

a) Show that $\Gamma_t \exp(-\int_0^t \beta_s \mathrm{d}s)$ is a local martingale;

b) Find a probability measure $\mathbb{Q}_T$ under which $(\Gamma_t)_{t\in[0,T]}$ is a local martingale;

c) Compute $\mathrm{d}\Gamma_t^{-1}$.

d) Find a probability measure $\mathbb{R}_T$ under which $(\Gamma_t^{-1})_{t\in[0,T]}$ is a local martingale;

Exercise 3. [Pts 2+2+2+2] *(Variation of constants)* Consider the nonlinear SDE

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + c(t)X_t \mathrm{d}B_t, \qquad X_0 = x,$$

where $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c: \mathbb{R}_+ \to \mathbb{R}$ are continuous deterministic functions.

a) Find an explicit solution Z_t in the case $f = 0$ and $Z_0 = 1$.

b) Use the Ansatz $X_t = C_t Z_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

c) Apply this method to solve the SDE

$$\mathrm{d}X_t = X_t^{-1}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x$$

where α is a constant.

d) Apply the method to study the solution of the SDE

$$\mathrm{d}X_t = X_t^\gamma \mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x > 0$$

where α and γ are constants. For which values of γ do we get explosion?