Note 1 (v.1 – May 6th 2020)

Quantum phenomenology and the mathematical model

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We start this course by describing one of the experiments which led to some of the early discoveries in quantum mechanics, that of the quantisation of the intrinsic magnetic moment of the electron, the spin.

1 The Stern–Gerlach experiment

Otto Stern and Walther Gerlach conducted in Frankfurt in 1922 the experience described in Figure 1 (left). A beam of atoms experience an intense magnetic field and as a consequence is deflected. Upon detection by means of a screen the arrival positions of the atoms reveals a quantized patterns, in contrast with classical theory of the magnetic moment of atoms which would require a continuous distribution of arrival positions due to the uniform distributions of the magnetic moment within the atom's population escaping from the oven. Figure 1 (right) shows the actual images obtained in the original experiment.

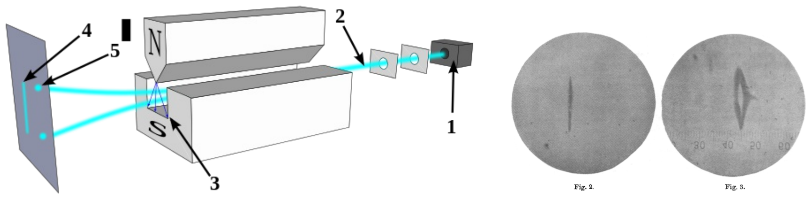
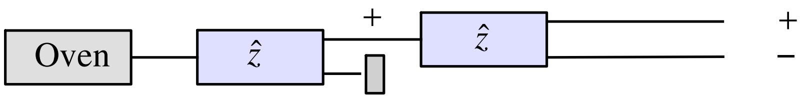


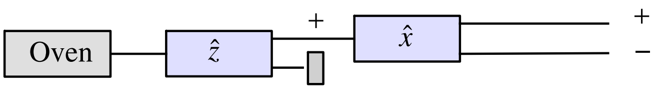
Figure 1. Left: Stern–Gerlach experiment: silver atoms travel through an inhomogeneous magnetic field and are deflected up or down depending on their spin. 1: furnace. 2: beam of silver atoms. 3: inhomogeneous magnetic field. 4: expected result. 5: what was actually observed. [from Wikipedia https://en.wikipedia.org/wiki/Stern–Gerlach_experiment]. Right: the experimental result of the Stern-Gerlach experiment. The beam has split into two components. From [Gerlach, Walther, and Otto Stern. "Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld." *Zeitschrift für Physik* 9, no. 1 (December 1, 1922): 349–52. https://doi.org/10.1007/BF01326983.]

The Stern–Gerlach experiment shows the quantisation of the magnetic moment for the electron. Indeed the silver atoms have atomic number 47. In its fundamental state, 46 of these electrons do not contributed to the magnetic moment since they come in pairs of opposite intrinstic magnetic moment (*spin*) and in a spatially symmetric state which do not generate any angular momentum. Only the last electron, whose spatial distributions is also symmetric, has an uncompensated intrinsic magnetic moment which consitute the only relevant contribution to the total magnetic moment of the atom. This magnetic moment interacts with the non–uniform magnetic field deflecting the trajectory of the atom. The presence of two well separated tracks means that this spin comes only in two varieties, oriented in the direction of the magnetic field or in the opposite direction.

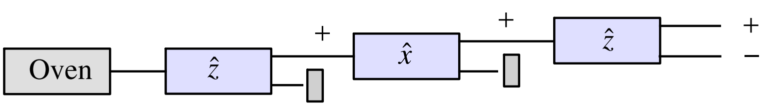
So the spin of the electron is a Bernoulli random variable. In order to explore other properties of this random variable we imagine a sequence of Stern–Gerlach experiments performed in series.



In this first case we first measure the $\hat{z}$ orientation, select those atoms which emerge from the + path after the first instrument and then again the $\hat{z}$ orientation and we obtain that all the atoms emerge from the + path.

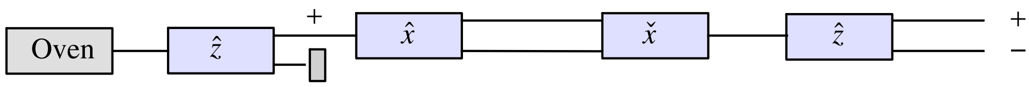


In this second situation we measure a different, orthogonal direction in the second instrument and we obtain that half of the atoms emerge from the + path and half from the – path. This is expected due to the symmetry of the problem.

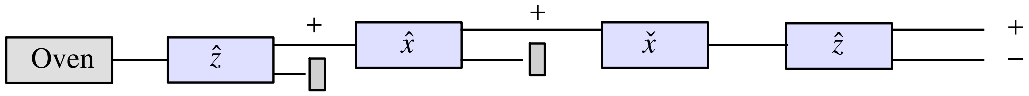


In this third installment we select the atoms which emerge from the + path after the $\hat{x}$ instrument and perform another selection with a $\hat{z}$ instrument. The result is that again half of the atoms emerge from the + path and half from the − path. The interpretation is that the measurement of $\hat{x}$ has completely destroyed the previous measurement of $\hat{z}$.

We now introduce another apparatus which undo the effect of a Stern–Gerlach instrument, this is not difficult to imagine, we just need to produce the opposite magnetic field to undo the effect of the first and arrange appropriately the geometry to recombine the atom beam. We label this instrument $\check{z}$ if it operates in the z direction.



In this first case we use the new instrument to recombine the beams after a $\hat{x}$ beam splitter. If we have selected only atoms with spin in the $\hat{z} = +1$ direction right after the oven, then we will end up with all the atoms in the + beam after the last $\hat{z}$ instrument.



We now block the $\hat{x} = -1$ beam and we observe that atoms exit the instrument with probability $1/2$ in each of the two final beams.

This is quite surprising. Allowing *more* atoms to go through the experiment depletes one of the exit beams! This property is not in agreement with a probabilistic description of the state of the atoms. Removing a conditioning cannot renders impossible events which were possible under the conditioning. This is a manifestation of quantum mechanical *interference* effects.

2 The mathematical model of a physical system

We describe now the basic mathematical model for a physical system from which we will later deduce the basic structure of quantum mechanics.

For a more systematic discussion of various aspects of this modelling step refer to the following literature:

- Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.
- Segal, E. E., and George Whitelaw Mackey. *Mathematical Problems of Relativistic Physics*. Providence, RI: American Mathematical Society, 1963.

We have two basic players in this game: observables and states.

- *Observables*. An observable is a physical quantity which we can measure (e.g. components of magnetic moment, position, speed/momentum, energy). Connected with some measuring apparatus which has a scale where you read a real number. We write $\mathcal{O}$ for the set of all observables. Given an observable $A \in \mathcal{O}$ more observables can be constructed from A by elementary procedures (i.e. relabeling the scale of the apparatus) E.g. $\lambda A, A^n \in \mathcal{O}$ $\lambda \in \mathbb{R}$. $A^n A^m = A^{n+m}$. In general we could imagine to define in a similar way $f(A)$ for any $f: \mathbb{R} \to \mathbb{R}$. An observable is *positive* if gives only positive results, in symbols we can reformulate this property as $A \geqslant 0 \Leftrightarrow \exists B \in \mathcal{O}: A \equiv B^2$ (there with $\equiv$ we just mean that operationally the two observables A and B^2 gives the same values).

- *States*. We imagine that a certain physical object under study can be *prepared* in such a way that it is meaningful to speak about repeated experiments on the *same* entity. This entity is the *state* $\omega \in \mathscr{S}$ of the system under consideration. E.g. the state of the atoms in the Stern–Gerlach experiment beam, the state of a particle in motion in a particle accellerator. (And what about "the state of world"?) There is a relation between measurements on states and values of observables and it is "statistical" in the sense that $\omega(A) = \langle \omega, A \rangle \in \mathbb{R}$ represent the measuring of A on the state ω, has to be considered as an average over "experiences". Operationally we measure an observable A in a given state ω by perfoming a sequence of repeated experiments and taking the average

$$\omega(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} m_{\omega}^{(i)}(A),$$

where each $m_{\omega}^{(i)}(A)$ is the i-th measurament of A in the state ω. A state is a map $\omega \colon \mathscr{O} \to \mathbb{R}$ understood as all the values it takes on every possible observable $\omega \equiv \{\omega(A) \colon A \in \mathscr{O}\}$.

We have the following relations between states and observables. You know that different states exists because when we measure an observable we get different numbers:

$$\omega(A) = \omega'(A), \forall A \in \mathscr{O} \Leftrightarrow \omega = \omega'.$$

You know that two observables are different because there is a state where they give different values:

$$\omega(A) = \omega(B), \forall \omega \in \mathscr{S} \Leftrightarrow A = B.$$

With respect to the operations we defined on observable we obtain the followin relations:

$$\omega(\lambda A) = \lambda \omega(A), \qquad \omega(A^n + A^m) = \omega(A^n) + \omega(A^m).$$

$$\omega(A^0) = 1 \Rightarrow A^0 = 1, \omega(1) = 1.$$

An observable is positive iff its value on any state is positive:

$$A \geqslant 0 \Leftrightarrow A = B^2 \Leftrightarrow \forall \omega \colon \omega(A) = \omega(B^2) \geqslant 0.$$

Therefore states are positive and normalized linear functionals on $\mathscr{O}$. We introduce a norm on $\mathscr{O}$ which measure the size of an observable $A \in \mathscr{O}$ via the largest possible value of a state on it:

$$\|A\| = \sup_{\omega \in \mathscr{S}} |\omega(A)|$$

Then

$$\|\lambda A\| = |\lambda| \|A\|, \quad \|A\| = 0 \Rightarrow A = 0.$$

We have also

$$\|A^2\| = \|A\|^2.$$

Indeed

$$\omega(\|A\| \pm A) = \|A\| \pm \omega(A) \geqslant 0 \text{“}\Rightarrow\text{”} \|A\| \pm A \geqslant 0.$$

$$\|A\|^2 - A^2 = (\|A\| + A)(\|A\| - A) \geqslant 0 \Rightarrow \omega(\|A\|^2 - A^2) \geqslant 0 \Rightarrow \|A\|^2 - \omega(A^2) \geqslant 0.$$

On the other hand

$$0 \leqslant (\|A\| \pm A)^2 = \|A\|^2 + A^2 \pm 2\|A\|A \Rightarrow 2\|A\||\omega(A)| \leqslant \|A\|^2 + \omega(A^2) \leqslant \|A\|^2 + \|A^2\|$$

taking sup over ω in $2\|A\||\omega(A)| \leqslant \|A\|^2 + \|A^2\|$ we get $\|A\|^2 \leqslant \|A^2\|$.

The states induce a linear structure over $\mathcal{O}$: we can define a new observable C by doing

$$\omega(C) = \omega(A) + \omega(B),$$

for given $A, B \in \mathcal{O}$. We can extend $\mathcal{O}$ to a linear space and

$$\|A + B\| \leqslant \|A\| + \|B\|.$$

So at this point if we assume completeness we will have a Banach space, but we are still not accounting for sequentials measurements. What about AB? Is not possible to define this using the previous arguments (i.e. via duality with states) if the observables are not simultaneously measurable (think about position and speed of a ball or frequency and duration of a musical note). If you cannot measure them simultaneously, then you cannot recover $\omega(AB)$ from $\omega(A)$ and $\omega(B)$.

It is reasonable to postulate that a **physical system** is defined by the set of its observable endowed with the operation of multiplication with scalars, addition and squaring (as we discussed above). That is to say that two physical systems are to be considered equivalent if their set of observables can be mapped one onto the other while preserving these structures. On such a structure one can define a notion of product (not associative in genera) via

$$A \circ B = \frac{1}{2}[(A + B)^2 - A^2 - B^2].$$

See in the book of Strocchi the discussion on this point at page 19, working with *Jordan algebras*. In order to obtain a well behaved mathematical theory we will introduce now an assumption which, while compatible with the previous discussion, cannot be justifyied on empirical ground.

Crucial techinical assumption. $\mathcal{O} \subseteq \mathcal{A}$ where $\mathcal{A}$ is a (non-commutative) algebra over $\mathbb{C}$ with involution $A \mapsto A^*$ and such that the following properties are true

$$(\lambda A + \beta B)^* = \bar{\lambda} A^* + \bar{\beta} B^*, \qquad (AB)^* = B^* A^*$$

$$\forall A \in \mathcal{A}, \quad A^* A \geqslant 0, \qquad \omega(A^* A) \geqslant 0 \qquad \omega \in \mathcal{S}$$

$$\|AB\| := \sup_{\omega \in \mathcal{S}} |\omega(AB)| \leqslant \|A\| \|B\|. \qquad \boldsymbol{\|A^* A\| = \|A\| \|A^*\|}.$$

One simple consequence: take $\lambda \in \mathbb{R}$

$$0 \leqslant \omega((\lambda A + 1)^* (\lambda A + 1)) = \lambda^2 \omega(A^* A) + \lambda \omega(A^*) + \lambda \omega(A) + 1$$

then $\omega(A^*) = \overline{\omega(A)}$ and from this we have $\|A^*\| = \|A\|$.

These technical assumptions can be implemented by assuming the following more coincise setting:

- Observables form a C^*-*algebra* $\mathscr{A}$ *with unity*.
- States $\mathscr{S}$ are normalized positive linear functionals on $\mathscr{A}$. We assume the set of states to be *full* (i.e. it separates the observables). Moreover observables should separate states (but this is by definition). Usually $\mathscr{S}$ is only a subset of all the positive linear functionals.

Example. Classical mechanical system $(q,p) \in \Gamma \subseteq T^*\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ where q is position and p momentum. The set of observables are the (continuous) functions $\mathscr{A} = C(\Gamma, \mathbb{C})$ $f^*(q,p) = \overline{f(q,p)}$. The states are (a subset of) the probability measures on Γ:

$$\omega(f) = \int_\Gamma f(q,p)\,\omega(\mathrm{d}q \times \mathrm{d}p).$$

$$\|f\| = \sup_{\omega \in \mathscr{S}} |\omega(f)|.$$

In classical physics one assume that states of the form $\omega = \delta_{(q_0,p_0)}$ are possible, these states are characterised by the fact that the *dispersion*

$$\Delta_\omega(f) = [\omega(f^2 - \omega(f)^2)]^{1/2} \geqslant 0,$$

is zero for all observables.

3 C^*-algebras

We discuss now the implication of the basic assumption on a physical system.

Definition 1. *A C^*-algebra $\mathscr{A}$ is an associative algebra over $\mathbb{C}$ which is endowed with the following additional structures: a norm $\|\cdot\|$ for which $\mathscr{A}$ is complete and which satisfy $\|ab\| \leqslant \|a\|\,\|b\|$ for all $a, b \in \mathscr{A}$ and an antilinear involution $*: \mathscr{A} \to \mathscr{A}$ for which $(ab)^* = b^*a^*$. These structures satisfy the following compatibility condition (C^* condition)*

$$\|a^*a\| = \|a\|^2, \qquad a \in \mathscr{A}.$$

We will usually denote by $1 = 1_{\mathscr{A}}$ the unity of $\mathscr{A}$. An element $a \in \mathscr{A}$ is *self-adjoint* if $a = a^*$, is *normal* if $a^*a = aa^*$, is unitary if $a^*a = aa^* = 1_{\mathscr{A}}$.

Note that $\|a\|^2 = \|a^*a\| \leqslant \|a^*\|\,\|a\|$, therefore $\|a^*\| = \|a\|$, i.e. the involution is isometric. Moreover $1^*a = (a^*1)^* = (a^*)^* = a$ and therefore $1^* = 1^*1 = 1$, from which follows $\|1\| = 1$.

Example 2. The algebra of all continuous complex-valued functions $C(X)$ on a compact space topological Hausdorff space X wrt. the pointwise product and endowed with the supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|, \qquad f \in C(X)$$

is a C^*-algebra.

Example 3. Let $\mathscr{H}$ be an Hilbert space. The set of all bounded linear operators $\mathscr{L}(\mathscr{H})$ on $\mathscr{H}$ together with the operator norm

$$\|A\| = \sup_{\varphi \neq 0} \frac{\|A\varphi\|}{\|\varphi\|}, \qquad A \in \mathscr{L}(\mathscr{H}),$$

and the involution given by the adjuction wrt. the scalar product of $\mathscr{H}$, is a C^* algebra, indeed by the property of the Hilbert space norm we have

$$\|A^*A\| = \sup_{\|\varphi\|=1} \|A^*A\varphi\| = \sup_{\|\varphi\|=\|\psi\|=1} \langle \psi, A^*A\varphi \rangle = \sup_{\|\varphi\|=\|\psi\|=1} \langle A\psi, A\varphi \rangle \leqslant \|A\|^2$$

and

$$\|A\|^2 = \sup_{\|\varphi\|=1} \|A\varphi\|^2 = \sup_{\|\varphi\|=1} \langle A\varphi, A\varphi \rangle = \sup_{\|\varphi\|=1} \langle \varphi, A^*A\varphi \rangle \leqslant \|A^*A\|.$$

Any norm-closed subalgebra $\mathscr{B}$ of $\mathscr{L}(\mathscr{H})$ which is self-adjoint, i.e. $\mathscr{B} = \mathscr{B}^*$ is a *concrete* C^*-*algebra*. For example, the compact operators form such a subalgebra or the C^*-algebra $C(T)$ generated by a single bounded self-adjoint operator T, i.e. the closure of all the polynomials in T, T^*, I.

Definition 4. *A Banach algebra $\mathscr{B}$ is a Banach space with a product such that $\|ab\| \leqslant \|a\| \, \|b\|$.*

Example 5. Take $L^1(\mathbb{R};\mathbb{C})$ with product given by convolution, then it is a Banach algebra. The same is true for $L^1(\mathbb{R}_+;\mathbb{C})$ with half-line convolution.

3.1 Spectral theory

In any (unital) Banach algebra $\mathscr{B}$ we can define the *spectrum* $\sigma(a) = \sigma_{\mathscr{B}}(a)$ of an element $a \in \mathscr{B}$ to be the set of $\lambda \in \mathbb{C}$ for which $(\lambda - a)$ is not invertible in $\mathscr{B}$. The complement of the spectrum is called the *resolvent set* and $R_a(\lambda) = (\lambda - a)^{-1}$ is the resolvent function.

Theorem 6. *For any $a \in \mathscr{B}$, the spectrum $\sigma(a)$ is a non-empty compact set and the resolvent function is analytic in $\mathbb{C} \backslash \sigma(a)$.*

Proof. For $|\lambda|$ large enough (i.e. $|\lambda| > \|a\|$) we have

$$R_a(\lambda) = (\lambda - a)^{-1} = \sum_{n \geqslant 0} \lambda^{-1-n} a^n \tag{1}$$

where the series is convergent in $\mathscr{B}$. This shows that $R_a(\lambda)$ is analytic with Laurent expansion at infinity and there $R_a(\lambda) \to 0$. On the other hand, if $\mu - a$ is invertible, we have the convergent series expansion

$$R_a(\lambda) = (\lambda - a)^{-1} = \sum_{n \geqslant 0} (\mu - a)^{-n-1} (\lambda - \mu)^n$$

valid in a neighborhood of $\mu \in \mathbb{C}$. So the resolvent is analytic in the complement of $\sigma(a)$ and $(\sigma(a))^c$ is an open set containing all $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$, therefore $\sigma(a)$ is compact. Assume that $\sigma(a)$ is empty. Then $R_a(\lambda)$ would be an entire function of λ which go to zero at infinity. As a consequence $R_a(\lambda)$ should be constant. Indeed this is true for $f(R_a(\lambda))$ for any continuous linear functional f and by Hanh-Banach this implies that $R_a(\lambda) = 0$. Therefore the spectrum must be non-empty. □

Proposition 7. *(Spectral radius formula) For any $a \in \mathscr{B}$ we have*

$$\rho(a) := \sup_{\lambda \in \sigma(a)} |\lambda| = \lim_{n\to\infty} \|a^n\|^{1/n} \leqslant \|a\|$$

with equality in case of a normal element of a C^*-algebra.

Proof. Let $r = \inf_{n\to\infty} \|a^n\|^{1/n}$, then $r \leqslant \|a\|$. Take m such that $\|a^m\|^{1/m} \leqslant r + \varepsilon$, then

$$r \leqslant \liminf_{n\to\infty} \|a^n\|^{1/n} \leqslant \limsup_{n\to\infty} \|a^n\|^{1/n} = \limsup_{n\to\infty} \|a^{mk(n,m)+\ell(n,m)}\|^{1/n} \leqslant \limsup_{n\to\infty} \|a^m\|^{k(n,m)/n} \|a^{\ell(n,m)}\|^{1/n}$$

$$\leqslant \limsup_{n\to\infty} (r+\varepsilon)^{mk(n,m)/n} = r + \varepsilon.$$

Therefore the limit indeed exists and equality is true for normal elements since

$$\|a^2\|^2 = \|a^* a^* a a\| = \|a a^* a^* a\| = \|a^* a\|^2 = \|a\|^4$$

and therefore $\|a^{2^k}\| = \|a\|^{2^k}$ and $\|a\| = \lim_{k\to\infty} \|a^{2^k}\|^{2^{-k}} = \lim_{n\to\infty} \|a^n\|^{1/n}$. By the convergence of the resolvent series (1) we have that $r = \sup_{\lambda\in\sigma(a)} |\lambda|$. □

A linear functional $\varphi: \mathscr{B} \to \mathbb{C}$ is multiplicative if $\varphi(ab) = \varphi(a)\varphi(b)$. The space $\mathscr{B}^*$ of linear functionals on $\mathscr{B}$ is a Banach space with the norm $\|\varphi\| = \sup_{a\in\mathscr{B}, \|a\|\leqslant 1} |\varphi(a)|$. The weak-$*$ topology on $\mathscr{B}^*$ is the topology generated by the system of neighborhoods of the form

$$N_{\psi, a_1, \ldots, a_n, \varepsilon} = \{\varphi \in \mathscr{B}' : |\psi(a_i) - \varphi(a_i)| \leqslant \varepsilon, \quad i = 1, \ldots, n\}$$

for $\psi \in \mathscr{B}'$, $a_1, \ldots, a_n \in \mathscr{B}$, $\varepsilon > 0$. It is the coarsest topology for which the maps $\varphi \in \mathscr{B}' \mapsto \hat{a}(\varphi) = \varphi(a) \in \mathbb{C}$ are continuous for all $a \in \mathscr{B}$. The Banach-Alaoglu theorem ensure that the closed unit ball of $\mathscr{B}^*$ is compact for the weak-$*$ topology.

Lemma 8. *The space $\Sigma(\mathscr{B})$ of all the multiplicative linear functionals on $\mathscr{B}$ is a compact Hausdorff space when endowed with the weak-$*$ topology.*

Proof. Let $\varphi \in \Sigma(\mathscr{B})$. Assume that $1 = \varphi(a) > \|a\|$, let b be the solution to $b = 1 + ab$ which exists since $\|a\| < 1$, then $\varphi(b) = \varphi(1) + \varphi(a)\varphi(b)$ which implies $\varphi(1) = 0$ and therefore $\varphi(a) = \varphi(a1) = \varphi(a)\varphi(1) = 0$ giving a contradiction. Therefore we have $|\varphi(a)| \leqslant \|a\|$, that is φ is continuous. On the space of all linear functionals we can consider the norm $\|\varphi\| = \sup_{\|a\|=1} |\varphi(a)|$ and obtain that $\|\varphi\| = 1$. Therefore by Banach-Alaoglu the unit ball is weakly-$*$ compact. Limits of multiplicative functionals are multiplicative, so $\Sigma(\mathscr{B})$ is also compact. □

For any $a \in \mathscr{B}$ we can define a continuous function $\hat{a}: \Sigma(\mathscr{B}) \to \mathbb{C}$ as $\hat{a}(\varphi) = \varphi(a)$, it is called the Gelfand transform of a. The function is continuous by the definition of the weak-$*$ topology on linear functionals.

Theorem 9. *The Gelfand transform is a contractive algebra homomorphism from $\mathscr{B}$ to $C(\Sigma(\mathscr{B}))$. The image algebra separates the points of $\Sigma(\mathscr{B})$.*

Proof. We have $\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \hat{a}(\varphi)\hat{b}(\varphi)$ and $\|\hat{a}\| = \sup_{\varphi \in \Sigma(\mathscr{B})} |\hat{a}(\varphi)| \leqslant \|a\|$. For any two points $\varphi \neq \psi$ in $\Sigma(\mathscr{B})$ there exists $a \in \mathscr{B}$ such that $\varphi(a) \neq \psi(a)$, therefore $\hat{a}(\varphi) \neq \hat{a}(\psi)$. □

For commutative Banach algebra $\mathscr{B}$ any proper maximal ideal is closed and any proper ideal is contained in a proper maximal ideal. Moreover let $\mathscr{I}$ a proper maximal ideal, then the quotient $\mathscr{B} \backslash \mathscr{I}$ is a Banach algebra and any $a \in \mathscr{B} \backslash \mathscr{I}$ is invertible, since otherwise $(a + \mathscr{I})\mathscr{B}$ would be a proper ideal containing $\mathscr{J}$. But a Banach algebra where any element is invertible must be $\mathbb{C}$ (Gelfand–Mazur theorem) so $\mathscr{B} \backslash \mathscr{I} = \mathbb{C}$ and $\mathscr{J}$ is of codimension 1.

Remark 10. (Gelfand–Mazur) Assume that all the elements except 0 of a Banach algebra $\mathscr{B}$ are invertible, then take $a \in \mathscr{B}$ and $\lambda \in \sigma(a)$. Since $\lambda - a$ is assumed to be not invertible we must have $\lambda - a = 0$ and therefore $a = \lambda$. That is $\mathscr{B} = \mathbb{C}$.

A consequence is:

Corollary 11. *Assume $\mathscr{B}$ is commutative. If $a \in \mathscr{B}$ is invertible iff $\hat{a} \in \Sigma(\mathscr{B})$ is invertible, that is $\hat{a}(\varphi) \neq 0$ for all $\varphi \in \Sigma(\mathscr{B})$. Therefore $\sigma(a) = \sigma(\hat{a}) = \{\varphi(a): \varphi \in \Sigma(B)\}$ and $\sup\{|\lambda|: \lambda \in \sigma(a)\} = \|\hat{a}\|_\infty$.*

Proof. If a is invertible then $1 = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$ so $(\hat{a})^{-1} = \widehat{(a^{-1})}$ and therefore $\hat{a}(\varphi) \neq 0$ for all φ. If a is not invertible, then $a\mathscr{B}$ is a proper ideal of $\mathscr{B}$ since $1 \notin a\mathscr{B}$. Let $\mathscr{I}$ be a maximal proper ideal containing $a\mathscr{B}$ and let $\varphi = 0$ on $\mathscr{I}$ and $\varphi(1) = 1$. Then φ is multiplicative and $\hat{a}(\varphi) = 0$. To prove that $\sigma(\hat{a}) = \{\varphi(a): \varphi \in \Sigma(B)\}$ observe that if $\lambda \in \sigma(\hat{a})$ then $\lambda - a$ is not invertible and $\mathscr{J} = (\lambda - a)\mathscr{B}$ is an ideal contained in a maximal ideal. If φ is the corresponding linear functional then $\varphi(\lambda - a) = 0$ since obviously $\lambda - a \in \mathscr{I}$ and therefore $\varphi(a) = \lambda$. So for any $\lambda \in \sigma(a)$ there is a multiplicative φ for which $\lambda = \varphi(a)$. So $\sigma(\hat{a}) \subseteq \{\varphi(a): \varphi \in \Sigma(B)\}$. On the other hand if $\varphi(\lambda - a) = 0$ then $(\lambda - a)\mathscr{B} \subseteq \ker(\varphi)$ therefore $\lambda - a$ cannot be invertible because otherwise if $(\lambda - a)^{-1}$ exists then $1 = (\lambda - a)^{-1}(\lambda - a) \in (\lambda - a)\mathscr{B} \subseteq \ker(\varphi)$ so $\ker(\varphi)$ cannot be a proper ideal. □

Example 12. For $L^1(\mathbb{R}; \mathbb{C})$ with product given by convolution the Gelfand transform is the Fourier transform. For $L^1(\mathbb{R}_+; \mathbb{C})$ with half-line convolution the Gelfand transform is the Laplace transform.

In the case of C^* algebras we have an isomorphism $\mathscr{A} \approx C(\Sigma(\mathscr{A}))$ of C^* algebras.

Theorem 13. (Gelfand–Naimark) *Any abelian C^*-algebra $\mathscr{A}$ is isometrically isomorphic to $C(\Sigma(\mathscr{A}))$.*

Proof. We need to check the correct behaviour of the involution, that is $\varphi(a^*) = \overline{\varphi(a)}$. However note that if a is self-adjoint, then $U_t = e^{iat}$ is unitary and therefore $|\varphi(U_t)| \leqslant \|U_t\| = 1$. Moreover by the multiplicative property

$$1 \geqslant |\varphi(U_t)| = |\exp(i\varphi(a)t)| = \exp(-\mathrm{Im}(\varphi(a))t).$$

From which we deduce that $\varphi(a) \in \mathbb{R}$. Decomposing any $a \in \mathscr{A}$ as $a = b + ic$, with b, c self-adjoint, we obtain that

$$\varphi(a^*) = \varphi(b - ic) = \varphi(b) - i\varphi(c) = \overline{\varphi(b) + i\varphi(c)} = \overline{\varphi(a)},$$

that is $\widehat{a^*} = \bar{\hat{a}}$. Remember that for C^*-algebras we have that if a is normal then $\|a\|_{\mathscr{A}} = \varrho_{\mathscr{A}}(a)$ therefore we have

$$\|a\|_{\mathscr{A}} = \|\hat{a}\|_{C(\Sigma(\mathscr{B}))}.$$

Now use againg the C^*condition to get for any $a \in \mathscr{A}$ (observe that a^*a is self-adjoint)

$$\|a\|^2 \underset{C^*}{=\!=} \|a^*a\| = \|\widehat{a^*a}\| = \|\widehat{a^*}\,\hat{a}\|_\infty = \|\bar{\hat{a}}\,\hat{a}\|_{C(\Sigma(\mathscr{A}))} \underset{C^*}{=\!=} \|\hat{a}\|^2_{C(\Sigma(\mathscr{A}))}$$

so we conclude that the transform is an isomorphism. It is one to one since if $\varphi(a) = \varphi(b)$ for all φ then $\varphi(a - b) = 0$ for all φ, this implies that $\|a - b\| = 0$. □

Remark 14. Multiplicative linear functionals in $\mathscr{B}$ corresponds to maximal proper ideals. See Strocchi for the details.

Exercise 1. Prove that the set of diagonal $n \times n$ matrices with complex entries is an abelian C^* algebra. Determine $\sigma(a)$ for any $a \in \mathscr{A}$ and $\Sigma(\mathscr{A})$.

The Gelfand–Naimark theorem allows a functional calculus on the normal elements of a C^* algebra.

If $a \in \mathscr{A}$ is normal, then the C^* algebra $C^*(a)$ (generated by $1, a, a^*$) is Abelian and therefore isomorphic to $C(\Sigma(C^*(a)))$ but $\varphi \in \Sigma(C^*(a))$ is uniquely determined by the value of $\varphi(a) \in \mathbb{C}$ since for any polynomial $p(a, a^*)$ we have $\varphi(p(a, a^*)) = p(\varphi(a), \overline{\varphi(a)})$. Then $\sigma(a) = \sigma(\hat{a}) = \{\varphi(a) : \varphi \in \Sigma(C^*(a))\}$ and $\Sigma(C^*(a)) = \sigma(a)$. This means that for any $f \in C(\sigma(a))$ there exists a unique $h \in C^*(a)$ such that $\hat{h} = f$ under the Gelfand transform map. In this case we write $h = f(a)$.

Is easy to see that $f(g(a)) = (f \circ g)(a)$, that $f(a)$ is self-adjoint if f is real, etc…

Observe that, since $C^*(f(a)) \subseteq C^*(a)$ for any continuous $f : \sigma(a) \to \mathbb{R}$ and normal a we have

$$\sigma(f(a)) = \{\varphi(f(a)) : \varphi \in \Sigma(C^*(a))\} = \{f(\varphi(a)) : \varphi \in \Sigma(C^*(a))\} = f(\sigma(a)).$$

With non-normal elements one has a similar relations, however not is such great generality. Let $a \in \mathscr{A}$ and $f(z) = \sum_{n \geqslant 0} c_n z^n$ be holomorphic in a neighborhood of $\sigma(a)$, then $f(a) = \sum c_n a^n$ is well defined and $\sigma(f(a)) = f(\sigma(a))$ (spectral mapping principle). This is easy to see for polynomials. An interesting case is $\sigma(a^{-1}) = (\sigma(a))^{-1}$.

Moreover $\sigma(ab)$ and $\sigma(ba)$ differ at most by $\{0\}$. Indeed let $\lambda \notin \sigma(ba)$ and let $c = (\lambda - ba)^{-1}$ then

$$(\lambda - ab)(1 + acb) = (\lambda + \lambda acb - ab - abacb) = (\lambda - ab + a(\lambda - ba)cb) = (\lambda - ab + ab) = \lambda$$

so $\lambda - ab$ is also inverible unless $\lambda = 0$.

If a is unitary (i.e $aa^* = 1 = a^*a$) then $\sigma(a) \subseteq \{z : |z| = 1\}$.

If a is self-adjoint then $\sigma(a)$ contains either $\pm\|a\|$: indeed recall that $\sigma(a)$ is compact. For selfadjoint a we have $\sigma(a) \subseteq \mathbb{R}$ and by compactness and the fact that $\varrho(a) = \|a\|$ we conclude that there exists $\lambda \in \mathbb{R}$ such that $|\lambda| = \|a\|$.

The continuous functional calculus for a self-adjoint element a can be developed also as follows. Consider the map $T : p \mapsto p(a)$ where p runs over complex polynomials. Then by the spectral mapping principle we have $\sigma(p(a)) = p(\sigma(a))$ and $\|p(a)\| = \varrho(p(a)) = \sup_{\lambda \in \sigma(a)} |p(\lambda)|$ so $\|p(a)\| = \|p\|_{C(\sigma(a))}$. By Stone-Weierstrass, polynomials are dense in $C(\sigma(a))$ (since $\sigma(a)$ is compact) and we have that T extends by continuity to a map on $T : C(\sigma(a)) \to C^*(a)$ moreover $T(f)T(g) = T(fg)$, $T(f)^* = T(\bar{f})$ and $\|T(f)\| = \|f\|$ so it is an isomorphism of C^* algebras.

3.2 Positive elements

Definition 15. *We call $a \in \mathcal{A}$ positive if a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}_+$ and we denote with $\mathcal{A}_+$ the set of positive elements of $\mathcal{A}$ and also write $a \geqslant 0$.*

Some properties of positive elements have simple and clever proofs.

- If $a, b \in \mathcal{A}_+$ and $a + b = 0$ then $a = b = 0$. Indeed $\sigma(-b) = -\sigma(b) \in \mathbb{R}_+ \Rightarrow \sigma(b) = \{0\} \Rightarrow b = 0$. (use the spectral mapping principle and that $\|b\| = \varrho(b) = 0$). Now take $a + b + c = 0$ with $a, b, c \in \mathcal{A}_+$

- If a is self-adjoint and $\|a\| \leqslant 1$ then $a \in \mathcal{A}_+$ iff $\|1 - a\| \leqslant 1$. Indeed if $a \geqslant 0$ then $\sigma(1 - a) \subseteq [0, 1]$ and $\varrho(1 - a) = \|1 - a\| \leqslant 1$. Conversely $\|1 - a\| \leqslant 1, \|a\| \leqslant 1$ imply that $\sigma(a)$ is contained in the intersection of two balls of radius 1 centred in 1 and 0, that is $\sigma(a) \subseteq [0, 1] \subseteq \mathbb{R}_+$.

- $\mathcal{A}_+$ is a cone, i.e. $\lambda a \geqslant 0$ if $a \geqslant 0$ for all $\lambda > 0$. Moreover if $a, b \geqslant 0$ and $\|a\|, \|b\| \leqslant 1$ then let $c = (a + b)/2$ and observe that $\|c\| \leqslant 1$ and

  $$\|1 - c\| \leqslant \frac{1}{2}\|1 - a\| + \frac{1}{2}\|1 - b\| \leqslant 1$$

  so $c \geqslant 0$. Therefore $\mathcal{A}_+$ is closed convex cone. It is closed since if $a_n \to a$ and $a_n \geqslant 0$ then we can rescale the sequence in such a way to get $\sup_n \|a_n\| \leqslant 1$ and therefore $\|1 - a\| = \lim \|1 - a_n\| \leqslant 1$ and $\|a\| \leqslant 1$ so $a \geqslant 0$.

- By functional calculus every positive element has a positive square root $a^{1/2}$. It can be constructed as limit of polynomials in a (without constant term). Therefore the product of two commuting positive elements is positive. The positive square root is unique, indeed if $b, c \geqslant 0$ are such that $b^2 = c^2 = a$ we have that a, b, c commute among themselves and

  $$0 = (b^2 - c^2)(b - c) = b^3 + b^2c + c^2b - c^3 = (b - c)^2(b + c) \geqslant 0$$

so we need to have $(b-c)^2 b = (b-c)^2 c = 0$ so $0 = (b-c)^2 b - (b-c)^2 c = (b-c)^3$ and therefore $b = c$.

- By functional calculus one can decompose any self-adjoint element into the difference of two positive elements $a = a_+ - a_-$.

- Any element $a \in \mathscr{A}$ is the sum of four unitaries. Indeed write $a = p + iq$ with self-adjoint elements p, q and then assuming $\|p\|, \|q\| \leqslant 1$ consider the unitaties $p \pm i(1-p^2)^{1/2}$ and $q \pm i(1-q^2)^{1/2}$.

Note that ny $\mathscr{L}(\mathscr{H})$ any operator in the form A^*A is positive (i.e. it has positive spectrum) .

This was conjectured to be true also in abstract C^* algebras but Gelfand and Naimark could not prove it. The result was proven later by Kelley and Vaught and surprisingly the proof is quite nontrivial (but not long). We skip it, just register the fact that in a C^* algebra the following properties are equivalent:

$$1)\ a \in \mathscr{A}_+, \qquad 2)\ a = b^2, b = b^*, \qquad 3)\ \|1 - a/\|a\|\| \leqslant 1, \qquad 4)\ a = c^*c.$$

As we have seen property 3) implies that $\mathscr{A}_+$ is a closed convex cone. We say that $a \geqslant b$ if $a - b \in \mathscr{A}_+$.

Remark 16. If $a, b \geqslant 0$ then $a + b \geqslant 0$ however positivity is tricky due to non-commutativity. For example even if $0 \leqslant a \leqslant b$ it does not follow in general that $a^2 \leqslant b^2$ unless a, b commute. If we try to define $|a| = (a^*a)^{1/2}$ then is not true that $|a + b| \leqslant |a| + |b|$.

Let us give some true inequalities.

- We have $a \leqslant \|a\|$ and $a^2 \leqslant \|a\| a$ as easily seen from spectral consideations.

- $a \geqslant 0$ implies $cac^* \geqslant 0$ and by difference $a \geqslant b \Rightarrow cac^* \geqslant cbc^*$.

- $a \geqslant b \geqslant 0$ then $(\lambda - a)^{-1} \leqslant (\lambda - b)^{-1}$ for $\lambda \geqslant 0$. (see Meyer for a proof)

- $a \geqslant b \Rightarrow f(a) \geqslant f(b)$ for functions of the form $f(x) = x^\alpha$ with $\alpha \in (0, 1)$.

Let us note the following.

Proposition 17. *Let ω is a continuous linear functional such that $\|\omega\| = \omega(1) = 1$ then $\omega(a^*) = \overline{\omega(a)}$.*

Proof. We can assume that a is s.a. since then is easy to conclude. Assume that $\omega(a) = f + ig$ with $f, g \in \mathbb{R}$ I need to prove that $g = 0$. Take $a + ic$ with $c \in \mathbb{R}$ and observe that $(a + ic)^*(a + ic) = a^2 + c^2$ then $\omega(a + ic) = f + i(g + c)$ so

$$f^2 + (g + c)^2 = |\omega(a + ic)|^2 \leqslant \|a + ic\|^2 \underset{C^*}{=} \|(a + ic)^*(a + ic)\| = \|a^2 + c^2\| \leqslant \|a^2\| + c^2 \leqslant \|a\|^2 + c^2.$$

Now c is arbitrary so we get that $g^2 + 2gc \leqslant \|a\|^2$ which is impossible unless $g = 0$. □

3.3 States on C^* algebras

A linear functional on $\mathscr{A}$ is positive if $\omega(a) \geqslant 0$ for all $a \in \mathscr{A}_+$.

For positive linear functionals Cauchy-Schwarz inequality holds true:

$$|\omega(a^*b)|^2 \leqslant \omega(a^*a)\omega(b^*b).$$

Then we have

Proposition 18. *A linear functional $\omega \in \mathscr{A}^*$ is positive iff $\|\omega\| = \omega(1)$.*

Proof. Note that $\|a\| - a \geqslant 0$ then if ω is positive we have $\|a\|\omega(1) \geqslant \omega(a)$. On the other hand the Cauchy–Schwartz inequality holds for ω and

$$|\omega(a)|^2 \leqslant \omega(1)\omega(a^*a) \leqslant \omega(1)^2\|a^*a\| \leqslant \omega(1)^2\|a\|^2,$$

so ω is bounded and $\|\omega\| \leqslant |\omega(1)| = \omega(1)$. On the other hand if ω is bounded and $\|\omega\| = \omega(1)$ we can assume that $\omega(1) = 1$. Then for any $a \geqslant 0$ with $\|a\| = 1$ we have also $|\omega(1-a)| \leqslant \omega(1) = 1$ which implies $|1 - \omega(a)| \leqslant 1$ but since $\omega(a) = \omega(a^*) = \overline{\omega(a)}$ we have $\omega(a) \in \mathbb{R}$ and therefore $\omega(a) \geqslant 0$. □

Proposition 19. *Positive linear functionals separate $\mathscr{A}$ and $a \in \mathscr{A}_+$ iff $\omega(a) \geqslant 0$ for all positive linear functionals ω.*

Proof. Assume that $\omega(a) = 0$ for all positive ω. Decompose $a = b + ic$ with self-adjoint b, c. Then $\omega(b) = \omega(c) = 0$. But this implies that $\hat{b} = \hat{c} = 0$ and by Gelfand's isomorphism that $b = c = 0$ (recall that multiplicative functionals are bounded and therefore positive). Let us now prove the second part. If $\omega(a) \geqslant 0$ we have that $\omega(a) = \omega(a^*)$ and $\omega(a - a^*) = 0$. Since positive functionals separate $\mathscr{A}$ we must have $a = a^*$. But then taking ω to be multiplicative we deduce that $\sigma(a) \subseteq \mathbb{R}_+$, that is $a \in \mathscr{A}_+$. Let us prove the first part. □

Recall that a state isa normalized positive linear functional on $\mathscr{A}$. The set of positive linear functionals of norm $\leqslant 1$ is a compact convex closed set (in the weak-$*$ topology). By a theorem of Krein–Milman it is the closed convex hull of its extreme points which are called *pure states*. Recall that an extreme point of a convex set is a point which cannot be written as the convex combination of other points. Pure states separate points in $\mathscr{A}$.

Example 20. On $\mathscr{L}(H)$ the states given by $\omega(A) = \langle x, Ax \rangle$ for some normalized $x \in H$ are pure states.

3.4 The Gelfand–Naimark–Segal representation and the GN theorem

So far we conceptualized the basic structure of a physical system and the related observation and measurement theory (algebra of observables and the convex set of state of physical system). This applies both to classical and quantum (i.e. non-classical) systems. We also argued that a classical system is given by an algebra of observables given by continuous functions on a "state space". For the moment anything escaping this point of view will be *quantum* therefore we need to take a non-commutative algebra (by the Gelfand-Naimark theorem).

How do we do computations in a non-commutative C^* algebra? We need (concrete) representations of non-commutative C^*-algebras in order to use the theory to make prediction and compare to experiments.

The Gelfand–Naimark–Segal theorem allows to construct representations of C^* algebras on an Hilbert space starting from any state ω (i.e. normalized positive linear functional).

Namely we want to construct a map $\varphi: \mathscr{A} \to \mathscr{L}(H)$ for some complex Hilbert space H such that φ is linear, $\varphi(1) = 1$, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ where on the r.h.s. the involution is understood as the adjoint in the Hilbert space. This is also called a $*$-homomorphism.

Remark 21. Any multiplicative functional $\varphi: \mathscr{A} \to \mathbb{C}$ give a one-dimensional representation on the Hilbert space $H = \mathbb{C}$.

Let us observe that any such representation is necessarily a contraction. Indeed note that if $\lambda - a$ is invertible in $\mathscr{A}$ then exists $c \in \mathscr{A}$ s.t. $c(\lambda - a) = 1$ that implies $\varphi(c)(\lambda - \varphi(a)) = 1$ so $\lambda - \varphi(a)$ is also invertible, that is $\sigma_{\mathscr{L}(H)}(\varphi(a)) \subseteq \sigma_{\mathscr{A}}(a)$. So for C^*-algebras

$$\|\varphi(a)\|_{\mathscr{L}(H)} \underset{C^*}{=} \varrho_{\mathscr{L}(H)}(\varphi(a)) \leqslant \varrho_{\mathscr{A}}(a) \underset{C^*}{=} \|a\|.$$

If φ is an isomorphism (on his image), i.e. $\ker(\varphi) = \{0\}$ on has that φ is an isometry since φ^{-1} is another representation and $\|a\| = \|\varphi^{-1}(\varphi(a))\| \leqslant \|\varphi(a)\| \leqslant \|a\|$.

Each unit vector $x \in H$ give rise to a state $\omega: a \mapsto \omega(a) = \langle x, \varphi(a)x \rangle$ on $\mathscr{A}$ (generally not a pure one). We will see now that every state on $\mathscr{A}$ arises in this way.

Assume ω is a state and define the Hermitean form on $\mathscr{A}$:

$$\langle a, b \rangle_\omega = \omega(a^* b).$$

The linear space $\mathscr{A}$ with this scalar product is a pre-Hilbert space. Let

$$\mathscr{N} = \{a \in \mathscr{A}: \langle a, a \rangle = 0\}$$

the set of zero elements and define the Hilbert space $H_\omega = \overline{\mathscr{A} \backslash \mathscr{N}}$ where the bar denotes the completion wrt. the topology generated by the scalar product $\langle , \rangle_\omega$. Denotes $\|a\|_\omega = \langle a, a \rangle_\omega^{1/2}$ the corresponding norm. Observe that

$$\langle ba, ba \rangle_\omega = \omega(a^* b^* ba) \leqslant \|b^* b\| \omega(a^* a) = \|b\|^2 \omega(a^* a) = \|b\|^2 \langle a, a \rangle_\omega$$

so the operator $L_b: H_\omega \to H_\omega$ defined by $L_b a = ba$ on the dense subset $\mathscr{A}$ is bounded with norm $\|L_b\| \leqslant \|b\|$. Note that it is well defined, since $L_b a = 0$ if $a \in \mathscr{N}$. Moreover $L_b L_c = L_{bc}$ and $L_b^* = L_{b^*}$ as can be easily checked. Therefore $a \mapsto L_a$ is an homomorphism of C^* algebras (since $\{L_a: a \in \mathscr{A}\}$ is a C^* subalgebra of $\mathscr{L}(H_\omega)$), indeed recall that $\|L_b\|_{\mathscr{L}(H_\omega)}^2 = \|L_b^* L_b\|_{\mathscr{L}(H_\omega)}$. So $\varphi_\omega(a) = L_a$ is a representation of $\mathscr{A}$ on H_ω and if we denote by $\Omega_\omega = [1] \in \mathscr{H}_\omega$ we have $\omega(a) = \langle \Omega_\omega, L_a \Omega_\omega \rangle$.

Note that the set $\{L_a \Omega_\omega: a \in \mathscr{A}\} \subseteq H_\omega$ is dense in H_ω. Then one says that Ω_ω is a cyclic vector for the representation φ_ω and that the representation is cyclic.

If K is another Hilbert space supporting a cyclic representation $\pi\colon \mathscr{A} \to \mathscr{L}(K)$ with cyclic vector $\psi \in K$ such that $\omega(a) = \langle \psi, \pi(a)\psi \rangle_K$ then the map $a \in \mathscr{A} \mapsto \pi(a)\psi \in K$ is an densely defined isometry from H_ω to K since

$$\langle a, a\rangle_{H_\omega} = \omega(a^*a) = \langle \pi(a)\psi, \pi(a)\psi\rangle_K.$$

Therefore the cyclic representations of $\mathscr{A}$ associated to a state ω are unique up to isomorphism. In general one call it the GNS representation associated to the state ω.

- A state ω is *faithful* is $\omega(a^*a) = 0 \Rightarrow a = 0$. Which implies that $\|L_a\Omega_\omega\|_\omega = 0 \Rightarrow a = 0$. The GNS representation is faithful if $L_a = 0 \Rightarrow a = 0$ which is a weaker property.
- Consider the commutative setting and let $H = L^2(\Omega, \mathscr{F}, \mu)$ for some probability space $(\Omega, \mathscr{F}, \mu)$ then on this space there are three different C^* algebras acting with pointwise multiplication on the elements of H: that of the continuous functions (taking $\mathscr{F}$ to be the Borel σ-algebra on some compact space K), that of the measurable functions and that of the $L^\infty(\mu)$ functions (i.e. equivalence classes modulo μ-null sets).
- A measure μ on a compact space K with the Borel σ-algebra $\mathscr{F}$ gives a faithful representation of $C(K)$ if the support of μ is K. μ is never faithful on measurable function and by construction is faithful on $L^\infty(\mu)$.
- The space H_ω of the GNS construction can be thought as a non-commutative version of the commutative $L^2(\Omega, \mathscr{F}, \mu)$. However here right multiplication $R_b a = ab$ is not in general given by a bounded operator.
- And one cannot obtain faithful representations by quotienting (like in $L^\infty(\mu)$ because $\mathscr{N}$ is only a left ideal). This is however possible if the state is tracial, i.e. $\omega(ab) = \omega(ba)$.

Since states separate elements of $\mathscr{A}$ there are enough GNS representations to build a faithful representation of any C^* algebra, as stated by the (non-commutative) Gelfand–Naimark theorem.

Theorem 22. *(Gelfand–Naimark) The exists a faithful representation of $\mathscr{A}$ in Hilbert space H*

The Gelfand–Naimark theorem construct faithful representation of $\mathscr{A}$ in Hilbert space by a direct sum of the GNS representations over *all* the states.

Let $\mathscr{S}$ be the set of all the positive normalized states of $\mathscr{A}$ and consider the Hilbert space $H = \oplus_{\omega\in\mathscr{S}} H_\omega$ where the elements are (finite) families $x = (x_\omega)_{\omega\in\mathscr{S}}$ with $x_\omega \in H_\omega$, where the scalar product is

$$\langle x, x\rangle_H = \sum_{\omega\in\mathscr{S}} \langle x_\omega, x_\omega\rangle_{H_\omega}$$

and where $\varphi(a)x = (\varphi_\omega(a)x_\omega)_{\omega\in\mathscr{S}}$. Then φ is a isometric representation of $\mathscr{A}$ in $\mathscr{L}(H)$. (Gelfand–Naimark theorem)

Assume that $\varphi(a) = 0$. Then $0 = \|\varphi_\omega(a)\|^2_{H_\omega} = \omega(a^*a)$ for all $\omega \in \mathscr{S}$. However we have already seen that positive linear functionals separate elements of $\mathscr{A}$ so $a^*a = 0$ and $a = 0$. Therefore φ is injective and this implies that it is an isomorphism.

If $\mathscr{A}$ is separable is possible to take a countable subset of $\mathscr{S}$ to perform the construction, in this case H will become separable.

Then GN theorem shows that there is no loss of generality to consider representations of physical systems in Hilbert space.

Remark 23. Consider a state ω and a self-adjoint a such that ω is dispersion-free wrt. a, i.e. $0 = \Delta_\omega(a) = [\omega((a - \omega(a))^2)]^{1/2}$ then in the corresponding GNS representation we have

$$\omega((a - \omega(a))^2) = \langle \Omega_\omega, (\varphi(a) - \omega(a))^2 \Omega_\omega \rangle = \|(\varphi(a) - \omega(a))\Omega_\omega\|^2$$

so $(\varphi(a) - \omega(a))\Omega_\omega = 0$ and $\omega(a)$ is an eigenvalue of $\varphi(a)$ with eigenvector Ω_ω. In particular $\omega(a)$ should be in $\sigma(\varphi(a)) \subseteq \sigma(a)$.

3.5 Pure states and irreducible representations

We want now to discuss briefly the "simplest" representations.

Definition 24. *A representation φ on the Hilbert space H is irreducible if the only invariant subspaces of the family $(\varphi(a))_{a \in \mathscr{A}}$ are $\{0\}$ and H.*

Lemma 25. *The representation $\varphi: \mathscr{A} \to \mathscr{L}(H)$ is irreducible iff $\varphi(\mathscr{A})' = \mathbb{C}$ where $\mathscr{B}'$ denotes the commutant of the algebra $\mathscr{B} \subseteq \mathscr{L}(H)$, namely the set of $A \in \mathscr{L}(H)$ such that $[A, B] = 0$ for all $B \in \mathscr{B}$.*

Proof. If φ is reducible then let P be an orthogonal projection on a non-trivial invariant subspace. Then $\varphi(a)P = P\varphi(a)$ for all $a \in \mathscr{A}$ and therefore $P \in \varphi'(\mathscr{A})$. On the oher hand if $\varphi(\mathscr{A})' \neq \mathbb{C}$ then there exists a bounded self-adjoint $H \in \varphi(\mathscr{A})'$ and we can use it to construct a projector P by using the spectral calculus of H. Then $P \in \varphi(\mathscr{A})'$ and PH is a nontrivial invariant subspace of H. □

Note that an irreducible representation is necessarily cyclic wrt. any vector (otherwise there would be nontrivial invariant subspaces).

Proposition 26. *The GNS representation φ_ω is irreducible iff ω is extremal in the set of states, i.e. a pure state for $\mathscr{A}$.*

Proof. Assume φ_ω is reducible, then there exists a projection $P \in \varphi_\omega(\mathscr{A})'$. Note that $P\Omega_\omega \neq 0$ since otherwise $P\varphi(a)\Omega_\omega = \varphi(a)P\Omega_\omega = 0$ for all a and therefore $P = 0$. Similarly $(1 - P)\Omega_\omega \neq 0$. Then

$$\omega(a) = \langle \Omega_\omega, \varphi_\omega(a)\Omega_\omega \rangle = \langle P\Omega_\omega, \varphi_\omega(a)P\Omega_\omega \rangle + \langle (1-P)\Omega_\omega, \varphi_\omega(a)(1-P)\Omega_\omega \rangle$$

$$= \lambda\, \omega_1(a) + (1-\lambda)\, \lambda\, \omega_2(a)$$

with states $\omega_1(a) = \langle P\Omega_\omega, \varphi_\omega(a)P\Omega_\omega\rangle / \|P\Omega_\omega\|^2$, $\omega_2(a) = \langle(1-P)\Omega_\omega, \varphi_\omega(a)(1-P)\Omega_\omega\rangle / \|(1-P)\Omega_\omega\|^2$ and $\lambda = \|P\Omega_\omega\|^2 > 0$. So ω is not extremal. On the other hand if ω is not extremal then there exists ω_1 and $\lambda \in (0,1)$ such that $\omega(a^*a) \geqslant \lambda\,\omega_1(a^*a)$ so the bilinear form $\omega_1(b^*a)$ is continuous on H_ω and there exists a nontrivial bounded operator $T \in \mathscr{L}(H_\omega)$ such that $\omega_1(b^*a) = \langle b, Ta\rangle_\omega$. Then

$$\langle b, T\varphi_\omega(c)\,a\rangle_\omega = \omega_1(b^*ca) = \omega_1((c^*b)^*a) = \langle\varphi_\omega(c^*)b, Ta\rangle_\omega = \langle b, \varphi_\omega(c)Ta\rangle_\omega$$

for all $a, b \in \mathscr{A}$ and as a consequence $T\varphi_\omega(c) = \varphi_\omega(c)T$ so $T \in \varphi_\omega(\mathscr{A})'$. If $T = t \in \mathbb{C}$ then $\omega_1(b^*a) = t\omega(b^*a)$ and since $\omega_1(1) = \omega(1) = 1$ we have $t = 1$ so $\omega = \omega_1$, therefore $\varphi_\omega(\mathscr{A})' \neq \mathbb{C}$. □

Corollary 27. *A state ω on a commutative C^* algebra $\mathscr{A}$ is pure iff it is multiplicative.*

Proof. Note that the GNS construction give a one-dimensional Hilbert space H_ω in the commutative case, for any multiplicative ω since $\omega(a^*a) = |\omega(a)|^2$ so $\mathscr{A}\backslash\mathscr{N} = \mathbb{C}$ and the projection map is simply $a \mapsto \omega(a)$. Therefore $\varphi(\mathscr{A}) = \mathbb{C}$ and the representation is obviously irreducible. On any commutative algebra $\varphi(\mathscr{A}) \subseteq \varphi(\mathscr{A})' = \mathbb{C}$ so $\varphi_\omega(a)$ is a multiple of 1 by irreducibility we need to have $H = \mathbb{C}$, so $\omega(ab) = \langle\Omega_\omega, \varphi_\omega(a)\varphi_\omega(b)\Omega_\omega\rangle = \langle\Omega_\omega, \varphi_\omega(a)\Omega_\omega\rangle\langle\Omega_\omega, \varphi_\omega(b)\Omega_\omega\rangle = \omega(a)\omega(b)$ and ω is multiplicative. □

Therefore irreducible representations of commutative C^* algebras corresponds to multiplicative functionals (think why). And constitute the Gelfand spectrum of the algebra which, as we ahave see, can be thought of as a space where the algebra is represented as continuous functions. Any irreducible representation is therefore a point in this space and the element of the algebra act via point-wise multiplication.

In the non-commutative case and in concordance with the GN theorem, one think to the space of all irreducible representations as the equivalent “non-commutative space”. Indeed it is clear that pure states separate points and that they are labelled by the corresponding irreducible representation (because of cyclicity and of uniqueness of the GNS representation). However here the elements of the algebra acts in a more complex way on each “point” ω, namely as linear operators $\varphi_\omega(a)$ on the corresponding Hilbert space H_ω.