Lecture 10 – May 20th 2020 – 8:15 via Zoom – M. Gubinelli

1 Complementary observables in finite quantum system

Consider still systems with finitely many pure states. All the observables have to take only finitely many values, let say n. So we can assume that they have all the same spectrum with n points and to be given by

$$\Gamma = \{\gamma_k = e^{2\pi i k/n}\}_{k=0,\ldots,n-1}.$$

We want to construct an algebra of two non-commuting observables u, v where both have the same spectrum (as above) and they are complementary, and for that we mean here that we are trying to impose that $p^{v|u}_{\ell,k} = 1/n$ for any k, ℓ.

There is no loss of generality to restrict to operators in Hilbert space, they have to be unitary because $\Gamma \subset \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ and is clear we need at least a space of dimensions n otherwise we cannot accomodate the n different eigenvalues Γ. By abuse of language u, v the representatives of u, v in the space $\mathscr{L}(\mathbb{C}^n)$. Let $(\varphi_k)_k$ be the eigenvectors of u, i.e. $u\varphi_k = \gamma_k \varphi_k$ and then take $v\varphi_k := \varphi_{k+1}$ with $k+1$ understood modulus n. Now observe that $uv\varphi_k = u\varphi_{k+1} = \gamma_{k+1}\varphi_{k+1} = \gamma_{k+1} v\varphi_k = (\gamma_{k+1}/\gamma_k) vu\varphi_k$ for any $k = 0, \ldots, n-1$ so

$$uv = e^{2\pi i/n} vu. \tag{1}$$

If we assume that u, v generate the algebra of observables then this fixes the full algebraic structure. Observe also that $u^n = v^n = 1$.

Remark 1. Note that we could have defined $v\varphi_k = \alpha\varphi_{k+1}$ for some $\alpha \in \mathbb{S}$ and then we would have $v^n = \alpha^n$ and we could have put also $u\varphi_k = \beta\gamma_k\varphi_k$ for some $\beta \in \mathbb{S}$ and then we would have $u^n = \beta^n$. This preserves the commutation relation (1) but changes the spectra of u, v.

Remark 2. Observe also that (1) implies that $u^n v = v u^n$ and also $v^n u = u v^n$ so the elements u^n, v^n belongs to the center (i.e. the elements which commutes with all the others) of the algebra generated by u, v. If we assume that u, v generate each of them a maximally abelian subalgebra then we can conclude from the commutation relation only that $u^n, v^n \in \mathbb{C}$. From this one can see that any irreducible representation of the commutation relation is n dimensional.

In particular

$$0 = (\gamma_k^{-1} u)^n - 1 = (\gamma_k^{-1} u - 1) \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$$

and from this we deduce that $\pi^u_k := n^{-1} \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$ satisfies $u\pi^u_k = \gamma_k \pi^u_k$ so π^u_k is the orthogonal projection on the span of φ_k, indeed one can check that $(\pi^u_k)^* = \pi^u_k$ and $\pi^u_k \pi^u_\ell = \delta_{k,\ell}\pi^u_k$. So we have also $u = \sum_{k=0}^{n-1} \gamma_k \pi^u_k$. For v we can proceed in the same way and define π^v_k. Now let's compute $\sum_k \pi^u_k \pi^v_\ell \pi^u_k$ using (1) and get

$$\sum_k p^{v|u}_{\ell,k} \pi^u_k = \sum_k \pi^u_k \pi^v_\ell \pi^u_k = \frac{1}{n}, \qquad \ell = 1, \ldots, n-1$$

so as required we have $p^{v|u}_{\ell,k} = 1/n$. So we confirm that our choice of algebraic structure give indeed a maximally complementary pair of observables.

We want now to argue that u, v are sufficient to generate all $\mathscr{B}(\mathbb{C}^n)$ (i.e. all the $n\times n$ complex matrices). Let $X\in\mathscr{B}(\mathbb{C}^n)$ and observe that the operator

$$Y = \frac{1}{n^2}\sum_{k,\ell} u^{-k}v^{-\ell}Xv^{\ell}u^k,$$

satisfy $uY = Yu$ and $vY = Yv$ so Y commutes with all the algebra generated by u, v (this actually depends only on the commutation relation (1)). Then this means that Y is a multiple of the identity, because since it commutes with u we must have $Y = \sum_k y_k\pi_k^u$ but then $Y = vYv^* = \sum_k y_k v\pi_k^u v^* = \sum_k y_k\,\pi_{k+1}^u$ and this implies that $y_k = y_{k+1}$ that is $Y = \lambda\mathbb{1}$ for some $\lambda\in\mathbb{C}$ so for any X we have a $\lambda = \rho(X)$ and by thinking a bit is clear that $\rho\colon\mathscr{B}(\mathbb{C}^n)\to\mathbb{C}$ is a actually a positive linear functional (think about it, is clear from the definition of Y) and $\rho(\mathbb{1}) = 1$. The definition of Y implies easily that for any $X\in\mathscr{B}(\mathbb{C}^n)$

$$X = \sum_{k,\ell} u^k v^{\ell}\rho((u^k v^{\ell})^* X)$$

that is $(u^k v^{\ell})_{k,\ell}$ is an orthornomal basis of $\mathscr{B}(\mathbb{C}^n)$ with respect to the non-degenerate scalar product $\langle X, Y\rangle = \rho(X^*Y)$. So in particular the algebra generated by u, v span all the $n\times n$ complex matrices.

This proves that the representation we gave is irreducible and therefore the pure states of this algebra are exaclty the vector states of this representation. So to describe all the possible states is enough to restrict to states of the form

$$\omega(X) = \mathrm{Tr}_{\mathbb{C}^n}[\rho\,\pi(X)],$$

where $\rho\in\mathscr{B}(\mathbb{C}^n)$ is a density matrix (i.e. $\rho\geqslant 0$, $\mathrm{Tr}_{\mathbb{C}^n}(\rho) = 1$) and π is the concrete representation of this algebra that we have analyzed. So the pure states are those for which $\omega(X) = \langle\psi, \pi(X)\psi\rangle$ for some unit vector in $\mathbb{C}^n$, i.e. ρ has to be of rank one. All the pure states of this quantum system are described by a ray in $\mathbb{C}^n$ i.e the set $\{e^{i\theta}\psi\colon\theta\in\mathbb{C}, \|\psi\| = 1\}$. This is very different from the commutative case where two observables u, v with each n different values have has possible pure states the n^2 different values of the pair.

The ray ψ is called the wave-function of the system and it provides a complete description as we saw. However it is so only because it parametrize the set of all pure states. Irreducible representations are like “charts” that we use to compute over the manifold of all the possible states of a physical (quantum) system.

We have completely classified this quantum system.

2 Quantum degrees of freedom

Assume that $n = n_1 n_2$ for two integers n_1, n_2 then there exist an alternative way to construct two complementary set of observables which each of them is maximally abelian. For $\alpha = 1, 2$, make the same construction above with n_α and obtain $u_\alpha, v_\alpha\in\mathscr{B}(H_\alpha)$ on the space $H_\alpha = \mathbb{C}^{n_\alpha}$ and consider the Hilbert space product $H = \mathbb{C}^n = \mathbb{C}^{n_1}\times\mathbb{C}^{n_2}$ and let u_α, v_α act on this product in the natural way so that u_1 and v_1 commutes with u_2, v_2. The operator u_1, u_2 together generate an abelian subalgebra and is maximal. Same for v_1, v_2 moreover the monomials $u_1^{k_1}u_2^{k_2}v_1^{\ell_1}v_2^{\ell_2}$ generates all $\mathscr{B}(H)$, so this representation is irreducible. And by the same reasoning as above we can show that $p_{(k_1,k_2),(\ell_1,\ell_2)}^{(v_1,v_2)|(u_1,u_2)} = 1/n$, so these pairs of maximally commutative observables are complementary.

So the full system $\mathscr{B}(H)$ splits into two subsystems $\mathscr{B}(H_1)$ and $\mathscr{B}(H_2)$ which do not interefere with each other. They represent two physically *kinematically* independent quantum systems $\mathscr{A}_1, \mathscr{A}_2$ whose observable algebras are generated resp. by (u_1, v_1) and (u_2, v_2). They could be not really independent because like in classical probability independence is a notion linked to a state.

Remark 3. Note that there could exists pure states of the composite system which make the observables in two subsystems not independent, i.e. for which the pair u_1, u_2 is not a family of independent classical random variables. This is called *entaglement*. Maybe we will discuss it later.

So we can proceed that way for any n by factorising into prime factors. So we could think of as this construction when n is prime as giving very basic quantum systems.

Example, when $n = 2$ we have u, v satisfying $u^2 = v^2 = 1$ and $uv = -vu$. Let $\sigma_x = u$, $\sigma_y = v$, $\sigma_z = (-i)uv$ unitary and hermitian matrices for which we can check that they satisfy the commutation relations

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \qquad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2,$$

and moreover any 2×2 complex matrix X can be written $X = \alpha + \beta\sigma_x + \gamma\sigma_y + \delta\sigma_z$. The operators $(\sigma_x, \sigma_y, \sigma_z)$ are called Pauli matrices and describe a quantum degrees of freedom with only two possible values, i.e. the abelian subalgebras have a spectrum with two points. This is the kind of model suitable to model the Stern-Gerlach experiment.

Next week we will take the limit $n \to \infty$ and study the associated quantum system.