Lecture 11 – May 26th 2020 – 14:15 via Zoom – M. Gubinelli

The C^*algebras we introduced last week (let's call them discrete canonical pairs) gives examples of very simple and discrete quantum observables. In particular we could take a state on which u has a given value, meaning that there exist states ω_k such that $\omega_k(u^\ell) = e^{2\pi i \ell k/n}$ for all $\ell = 1, \ldots, n-1$ (recall that $u^n = 1$). These states are just given by

$$\omega_k(a) = \langle \varphi_k, a\varphi_k \rangle$$

where φ_k are the eigenfunctions of u. This means that ω_k is multiplicative on $C^*(u)$.

However we have also that it cannot be multiplicative on v (because u, v do not commue) and actually

$$\omega_k(v^\ell) = \langle \varphi_k, v^\ell \varphi_k \rangle = 0, \qquad \ell = 0, \ldots, n-1.$$

This means that they are uniformly distributed on the set $\{\exp(2\pi i k/n): k = 0, \ldots, n-1\}$.

Here their maximal complementary shows up in the fact that while one is completely determined, the other is uniformly distributed.

So in some sense they can be considered the quantum equivalent of discrete uniform random variables.

We would like now to take some limit $n \to \infty$ in order to produce in this way continuous analogs of these algebras. This would give us an example of non-commutative C^* algebra generated by two abelian subalgebras with continuous spectrum.

The intuition we want to carry on is how we go from discrete uniform r.v. to continuous ones. In particular imagine that X is a r.v. with continuous distribution described by a density $p(x)$ on $\mathbb{R}$. I can imagine to approximate it in law by taking a discrete r.v. X_L such that $X_L = [X]_L$ for $L \in \mathbb{N}$ where $[x]_L = \lfloor Lx \rfloor / L$. Then we have for any continuous and bounded function $f: \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[f(X_L)] = \int_{\mathbb{R}} f([x]_L) p(x) \mathrm{d}x \to \int_{\mathbb{R}} f(x) p(x) \mathrm{d}x = \mathbb{E}[f(X)].$$

Let's try to implement the same procedure for a C^*-algebra. The first observation is that if we denote (u_n, v_n) a discrete canonical pair of degree n we have the following. We can take $L^2(\mathbb{T})$ as Hilbert space where $\mathbb{T} = \mathbb{R} \backslash \mathbb{Z}$ and represent each u_n and v_n as

$$u_n f(x) = \exp(2\pi i\, [x]_n) f(x), \qquad v_n f(x) = f(x - 1/n), \qquad x \in \mathbb{T}.$$

One can check that u_n, v_n is a representation of the algebra we constructed above. In this way we can embed all the operators $(u_n, v_n)_{n \geqslant 0}$ into $\mathscr{B}(L^2(\mathbb{T}))$.

We have to understand what plays the role of “continuous functions” in this context. We just take monomials of the form $u_n^k v_n^\ell$ (they suffice to determine any other element of $C^*(u_n, v_n)$ due to their commutation relation). However is easy to see that $u_n^k v_n^\ell \to 1$ in the weak topology of $L^2(\mathbb{T})$. Somehow we need to look at high powers of u_n, v_n to see something interesting. We take $\ell_n = n^{1/2}[s]_{n^{1/2}}$ and $k_n = n^{1/2}[t]_{n^{1/2}}$ and now consider

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \overline{f_n(x)} \exp(2\pi i k_n [x]_n) g_n(x - \ell_n/n) \mathrm{d}x.$$

Note that we choose ℓ_n, k_n in that particular way since the commutation relations reads

$$u_n^{k_n} v_n^{\ell_n} = e^{2\pi i k_n \ell_n/n} v_n^{\ell_n} u_n^{k_n} = e^{2\pi i [s]_{n^{1/2}} [t]_{n^{1/2}}} v_n^{\ell_n} u_n^{k_n}$$

so the choice of the factor $n^{1/2}$ was due to the nice cancellation in the phase factor here. By rescaling we have, for functions f_n, g_n supported on $(-\pi,\pi)$ and letting $x = y/n^{1/2}$.

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n\rangle_{L^2(\mathbb{T})} = \int_{(-\pi,\pi)} \overline{f_n(x)} \exp(2\pi i [t]_{n^{1/2}} [x]_n n^{1/2}) g_n(x - [s]_{n^{1/2}}/n^{1/2}) \mathrm{d}x$$

$$= n^{-1} \int_{(-\pi n^{1/2}, \pi n^{1/2})} \overline{f_n(y/n^{1/2})} \exp(2\pi i [t]_{n^{1/2}} n^{1/2} [y/n^{1/2}]_n) g_n((y - [s]_{n^{1/2}})/n^{1/2}) \mathrm{d}y$$

so to have a well defined limit we can take $f_n(x) = n^{1/4} f(n^{1/2} x)$ and $g_n(x) = n^{1/4} g(n^{1/2} x)$ with $f, g \in C_0^\infty(\mathbb{R})$ so that for n large enough we have

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{R}} \overline{f(y)} \exp(2\pi i [t]_{n^{1/2}} n^{1/2} [y/n^{1/2}]_n) g(y - [s]_{n^{1/2}}) \mathrm{d}y$$

so here now we can take the limit and obtain that

$$\lim_n \langle f_n, u_n^{k_n} v_n^{\ell_n} g_n\rangle_{L^2(\mathbb{T})} = \langle f, U(t) V(s) g\rangle_{L^2(\mathbb{R})} \tag{1}$$

where (U, V) are two **unitary groups** acting on $L^2(\mathbb{R})$ as

$$U(t) f(y) = \exp(2\pi i t y) f(y), \qquad V(s) f(y) = f(y - s).$$

Unitary group means that $U(t)^* = U(-t)$, $U(t)U(s) = U(t+s)$ for all $t, s \in \mathbb{R}$ and $U(0) = 1$. These relations come from the formula for the convergence in law above.

Exercise 1. Justify that U, V are unitary groups. Actually try to prove it using only (1) and not the explicit form of the operators.

Moreover they are weakly continuous, i.e. $t \mapsto \langle f, U(t) g\rangle$ is continuous for all $f, g \in L^2(\mathbb{R})$. Since they are unitary they are also strongly continuous.

They satisfying the commutation relations

$$U(t) V(s) = e^{2\pi i s t} V(s) U(t), \qquad t, s \in \mathbb{R}. \tag{2}$$

These commutation relations are called the Weyl form of the canonical commutation relations and they are the implementation of the Heisenberg's commutation relations

$$[P, Q] = i\hbar,$$

within the C^*-framework (i.e. working only with bounded operators). The link between these formulas comes from interpreting the two unitary groups as being generated by the self-adjoint operators P, Q i.e. as

$$U(t) = \exp(iQt), \qquad V(s) = \exp(iPt),$$

and recalling that Baker-Campbell-Hausdorff forumula gives (under suitable conditions for unbounded self-adjoint operators A, B with $[A, B]$ given by a scalar that)

$$e^A e^B = e^{A + B + \frac{1}{2}[A, B]}.$$

Applying it formally to P, Q we have

$$e^{iQt} e^{iPs} = e^{i(Ps + Qt) + \frac{1}{2}[P, Q]} = e^{\frac{1}{2}[P, Q]} e^{i(Ps + Qt)}, \qquad e^{iPs} e^{iQt} = e^{i(Ps + Qt) - \frac{1}{2}[P, Q]} = e^{\frac{1}{2}[P, Q]} e^{i(Ps + Qt)}$$

so that

$$e^{iQt}e^{iPs} = e^{i\hbar st}e^{iPs}e^{iQt},$$

so in my notations $\hbar = 2\pi$.

Putting aside for the moment unbounded operators we obtained a pair of commutative C^* algebras $\mathcal{Q}, \mathcal{P}$ given by $\mathcal{Q} = C^*((U(t))_{t\in\mathbb{R}})$, $\mathcal{P} = C^*((V(s))_{s\in\mathbb{R}})$ which are concrete C^* algebras on $L^2(\mathbb{R})$. We denote $\mathcal{A} = C^*(\mathcal{Q}, \mathcal{P})$.

We will show that the spectrum of $\mathcal{Q}$ and $\mathcal{P}$ can be identified with a subset of $\mathbb{S} \subset \mathbb{C}$. So they are like random variables taking values on $\mathbb{S}$ and they can be easily parametrized by real number. In particular if ω is a state on $\mathcal{A}$ then the function $t \mapsto \omega(U(t))$ is continuous on $\mathbb{R}$ and positive definite and normalized so it corresponds to probability measure on $\mathbb{R}$, which we denote by $\mu^{\mathcal{Q},\omega}$ this is the law of $\mathcal{Q}$ on ω. Similarly for $\mathcal{P}$. However $\mathcal{Q}$ and $\mathcal{P}$ do not commute.

The C^*-algebra $\mathcal{A}$ is called the Weyl algebra. It is the fundamental example of two continuous observables which do not commute and in some sense they show complementarity.