Lecture 12 – May 27th 2020 – 8:15 via Zoom – M. Gubinelli

Canonical commutation relations

$$U(t)V(s) = V(s)U(t)\exp(2\pi ist), \qquad s,t \in \mathbb{R}. \tag{1}$$

Where $U, V$ are two unitary representations of the additive group of the reals, i.e.

$$U(t)U(s) = U(t+s), \qquad U(t)^* = U(-t),$$

and similarly for $V$.

### Unitary representations of ℝ and observables as homorphisms

For the moment let us concentrate on only one of the families, let's say $(U(t))_{t\in\mathbb{R}}$. I want to look at it at some kind of non-commutative Fourier transfrom (or characteristic function). It is giving me information about ab observable very much like the characteristic function give informations about a random variable.

Assume for the moment that we the family $(U(t))_{t\in\mathbb{R}}$ is a family of bounded operators on an Hilbert space $H$ (giving a representation of $\mathbb{R}$ on $H$).

For any unit vector $v \in H$ we can form the function $\varphi^v(t) = \langle v, U(t)v\rangle$, it is easy to show that $\varphi^v(0) = 1$, and $\varphi^v$ is positive definite, i.e.

$$\sum_{i,j} \bar{\lambda}_i \lambda_j \varphi^v(t_j - t_i) \geqslant 0 \qquad (\lambda_i)_i \subseteq \mathbb{C}, (t_i)_i \subseteq \mathbb{R}.$$

This are the same properties of the characteristic function of a measure, so we want to show that there exist a measure $\mu^v$ on $\mathbb{R}$ so that

$$\varphi^v(t) = \int_{\mathbb{R}} e^{itx} \mu^v(\mathrm{d}x), \qquad t \in \mathbb{R}.$$

This is essentially Bochner's theorem (given some continuity of $\varphi^v$), but are going to sketch a proof because will give us a simple example of more involved reconstruction we encouter later on.

A Radon measure on $\mathbb{R}$ is just a positive functional on $C_0(\mathbb{R})$ (Riesz–Markov–Kakutani).

Let $f \in \mathscr{S}(\mathbb{R})$ a Schwartz function and define

$$T_f := \int_{\mathbb{R}} U(t)\hat{f}(t)\mathrm{d}t$$

where $\hat{f}$ is the Fourier transform of $f$. In order for this definition to make sense I need some condition on the family $(U(t))_{t\in\mathbb{R}}$ to be able to integrate it. Is easy to check in simple cases that $(U(t))_{t\in\mathbb{R}}$ is essentially never continuous in the operator norm.

Indeed usually $\|U(t) - U(s)\| = 2$ for $t \neq s$, think about $V(s)f(x) = f(x-s)$ or $U(t)f(x) = e^{2\pi ix}f(x)$.

So the integral cannot be defined as Bochner integral. But it is reasonable to ask for weak measurability, in the sense that for any $v \in H$ we want $\varphi^v(t)$ to be measurable in $t$, by polarization this indeed implies that $\langle v, U(t)w\rangle$ is measureable for any $v, w \in H$. (Note indeed that $\langle v, U(t)w\rangle = \sum_{i=1}^4 \lambda_i \langle h_i, U(t)h_i\rangle$ for some four well chosen vectors $h_i$). So we can define $\langle v, T_f v\rangle := \int_{\mathbb{R}} \varphi^v(t)\hat{f}(t)\mathrm{d}t$ and polarization define the operator $T_f$. Note that it is a bounded operator because

$$|\langle v, T_f v\rangle| \leqslant \int_{\mathbb{R}} |\varphi^v(t)||\hat{f}(t)|\mathrm{d}t \leqslant \|v\|^2 \int_{\mathbb{R}} |\hat{f}(t)|\mathrm{d}t \leqslant C_f \|v\|^2$$

for all $f \in \mathscr{S}(\mathbb{R})$ and for all $f \in \mathscr{F}L^1 = \{f \in C(\mathbb{R}) : \|\hat{f}\|_{L^1} < \infty\}$. This define a linear functional $\ell^v$ on $\mathscr{S}(\mathbb{R})$ such that

$$|\ell^v(f)| \leqslant C_v \|\hat{f}\|_{L^1}.$$

In order to extend this functional to all $C_0(\mathbb{R})$ I need to show that $|\ell^v(f)| \lesssim \|f\|_\infty$ for $f \in \mathscr{S}(\mathbb{R})$. In order to do this one has to use that $\ell^v$ is positive, that is if $f = g^2 > 0$ then provided $g \in \mathscr{S}(\mathbb{R})$ we have

$$\ell^v(f) = \ell^v(g^2) \geqslant 0$$

because we use that

$$\int_{\mathbb{R}} \varphi^v(t) \hat{f}(t) \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi^v(t+s) \hat{g}(t) \hat{g}(s) \mathrm{d}t \mathrm{d}s \geqslant 0.$$

because we can approximate the two integrals by finite sums and use the positive definiteness of $\varphi^v$. Then one argue by approximation that for any $f \in \mathscr{S}(\mathbb{R})$ one has $\|f\|_\infty - f = h \geqslant 0$ and this can be approximated by $h_\varepsilon$ in $\mathscr{S}(\mathbb{R})$ to get that $\ell^v(h) \geqslant 0$ and this will imply that $\ell^v(f) \leqslant \|f\|_\infty$ and the the functional can be extended to all $C_0(\mathbb{R})$ by approximation. To make rigorous this argument one need that $\varphi^v$ is continuous in $t$.

As soon as we have extended $\ell^v$ continously we can define a $*$-representation $Q$ of $C_0(\mathbb{R})$ on $\mathscr{L}(H)$. For any $f \in C_0(\mathbb{R})$ define the operator $Q(f)$ by the relation $\langle v, Q(f) v \rangle = \ell^v(f)$ and its polarization. This define a bounded operator such that $\|Q(f)\|_{\mathscr{L}(H)} \leqslant \|f\|_\infty$ and $Q(f)^* = Q(\bar{f})$ and $Q$ is linear in $f$ and $Q(f)Q(g) = Q(fg)$ (by continuity is enough to check there relations of $f \in \mathscr{S}(\mathbb{R})$ and this case we have the more precise relation

$$Q(f) = \int_{\mathbb{R}} U(t) \hat{f}(t) \mathrm{d}t$$

(remember that the r.h.s is defined as a weak integral). I would like to use $f(x) = e^{isx}$, in order to do this observe that for any $v \in H$

$$\langle v, Q(f) v \rangle = \int_{\mathbb{R}} \varphi^v(t) \hat{f}(t) \mathrm{d}t,$$

looking at this formula is clear that if $f_n \to f$ in such a way that the r.h.s. converges, so we can take $f_n(x) = e^{isx} e^{-x^2/(2n)}$ so that

$$\langle v, Q(f_n) v \rangle = \int_{\mathbb{R}} \varphi^v(t) \hat{f}_n(t) \mathrm{d}t = (2\pi n^{-1})^{-1/2} \int_{\mathbb{R}} \varphi^v(t) e^{-n(t-s)^2/2} \mathrm{d}t \to \varphi^v(s)$$

by continuity of $\varphi^v$. So this suggest that we can define $Q(e^{is\cdot}) = U(s)$. In order for this to make full sense we need to extend $Q$ to all continuous bounded functions. Short way to do this is to realise that $\ell^v$ corresponds to a measure $\mu^v$ By Riesz-Markov and then just extend it using measure theory. In this case actually you can extend it to all bounded measurable functions on $\mathbb{R}$.

Note also that if $f_n \uparrow f$ then the sequence $(\langle v, Q(f_n) v \rangle)_n$ is monotone increasing since if $f \geqslant 0$ then $\langle v, Q(f) v \rangle \geqslant 0$ so we can extend $Q$ to all $C_b(\mathbb{R})$. To check that the extension is unique the following argument works.

Take now the family $(h_n(x) = \exp(-n x^2))_n$ then by continuity of $\varphi^v$ it is easy to prove that

$$Q(h_n) \to 1_{\mathscr{L}(H)}.$$

Observe that if $f \in C_b(\mathbb{R})$ then $h_n f \in C_0(\mathbb{R})$ and it follows that for any extension $Q'$ of $Q$ to $C_b(\mathbb{R})$ we have

$$Q'(h_n) Q'(f) = Q'(h_n f) = Q(h_n f) = Q(h_n) Q(f)$$

and taking limits we have $Q'(f) = Q(f)$.

So today we proved that for any weakly-continuous one-parameter unitary group in $\mathscr{L}(H)$ we can construct a representation $Q$ of the $C^*$-algebra $C_b(\mathbb{R})$ on $\mathscr{L}(H)$. It is suggestive to write $f(Q) = Q(f)$ and think to $f(Q)$ as a function computed on an operator $Q$ in such a way that the formula

$$U(t) = \exp(itQ)$$

has now a sense. We could of course associate to $Q$ an unbounded linear operator $\hat{Q}$ on a dense domain within $H$ in such a way that by Stone theorem $\hat{Q}$ is the generator of the group $(U(t))_{t\in\mathbb{R}}$.

From the operational point of view such an homomorphism $Q$ represent an observable in the sense that we can measure its expectation value on any state $\omega$ and also we can see it as a random variable with a law given by the linear functional

$$f \mapsto \omega(f(Q)).$$

If we go back to the Weyl relation we now understand that they describe two observables $P, Q$ which satisfy the commutation relations

$$\exp(itQ)\exp(isP) = \exp(2\pi ist)\exp(isP)\exp(itQ).$$

Combining unbouded operators is a task of the same difficulty of combining two homomorphism or two unitary representations of $\mathbb{R}$.

There is no simple way to understand, for example, the sum $P + Q$.

Tentantively in this course we take the attitude that an observable is really a *-homomorphism of $C_b(\mathbb{R})$ into either some abstract $C^*$-algebra or into a $C^*$-algebra of operators. This extends to the non-commutative/quantum context the probabilistic notion of real random variable.

This is coherent with our modelisation which sees observables as self-adoint elements of a $C^*$-algebra in that if $f: \mathbb{R} \to \mathbb{R}$ then $f(Q)$ is a self-adjoint operator.

The reason to use this different notion is that it can accomodate the case where we have dealing with "unbounded" obserables. Think for example to a Gaussian random variable $X$. A Gaussian random variable is not an element of a $C^*$-algebra since $X$ can take arbitrarily large values. However if we look at $X$ has a $*$-homomorphism by letting $X(f) := f(X)$ for any $f \in C(\mathbb{R})$ then $X$ is a well defined observable. In this case it has a concrete realisation on $L^2(\mathbb{P})$ and if we take $v(\omega) = 1$ we have that

$$\langle v, X(f)v\rangle = \mathbb{E}[f(X)].$$