Lecture 13 – June 2nd 2020 – 14:15 via Zoom – M. Gubinelli

Canonical commutation relations (Weyl form)

$$U(t)V(s) = V(s)U(t)\exp(ist), \qquad s,t \in \mathbb{R}. \tag{1}$$

Where U, V are two unitary representations of the additive group of the reals, i.e.

$$U(t)U(s) = U(t+s), \qquad U(t)^* = U(-t),$$

and similarly for V.

For the moment let's think of these objects a bounded operators on an Hilbert space H. So in particular they generate a concrete C^*-algebra with the norm given by the operator norm of the Hilbert space by closing the finite linear combinations of monomials in these operators wrt. the norm.

This C^*-algebra represents for us a single quantum degree of freedom in one dimenions (i.e. a one dimensional quantum particle).

The C^*-algebra structure is not enough to have a well-behaved theory of these operators, and we require that this concrete representation of the Weyl relations to be *regular*, meaning that the family U, V are strongly continuous wrt. the underlying Hilbert space (which is not a C^*-algebra notion).

In the last lecture we understood that to any strongly continuous family $(U(t))_{t\in\mathbb{R}}$ of operators in $\mathscr{L}(H)$ we can associate in essentially a unique way an $*$-homomorphism $Q: C(\mathbb{R}) \to \mathscr{L}(H)$ such that $Q(e^{i\alpha\cdot}) = U(\alpha)$ and similarly for V: $P: C(\mathbb{R}) \to \mathscr{L}(H)$ such that $P(e^{i\beta\cdot}) = V(\beta)$, then we can write $f(Q) := Q(f)$ and have that Weyl relations have the form

$$e^{i\alpha Q}e^{i\beta P} = e^{i\beta P}e^{i\alpha Q}e^{i\alpha\beta}.$$

Recall that $Q(f)$ is defined by polarisation of the relation $\langle v, Q(f)v\rangle = \ell^v(f)$ where v is any unit vector in H and ℓ^v is the unique positive functional on $C(\mathbb{R})$ which has appropriate locality properties (and therefore corresponds to a unique Borel probability measure μ^v on $\mathbb{R}$) and such that $\ell^v(e^{i\alpha\cdot}) = \langle v, U(\alpha)v\rangle$.

From the point of the of C^*algebraic approach the homomorphism Q, P represents families of observables which are then given by choosing a particular way f to measure the quantity Q so that we have a definite observable $Q(f)$, i.e. self-adjoint element of C^*. Let's call them *extended observables*.

If a is a self-adjoint element of a C^*-algebra $\mathscr{A}$ we can always via continuous functional calculus associate to it an observable A in this extended sense by letting $A(f) := f(a)$ and therefore have that $A \in \mathrm{Hom}(C(\mathbb{R}), \mathscr{A})$.

Extendend observables allows to handle quantities which are not naturally bounded and therefore cannot be represented by elements of the C^*-algebra.

Let's go back to the Weyl C^*-algebra (which can be defined without mentioning the Hilbert space representation, this will maybe be discussed by Panagiotis in a further talk).

For the moment we understand a Weyl C^*-algebra as given by the concrete realisation above (in particular regular).

Note that we can form the Weyl operators $(W(z))_{z\in\mathbb{C}}$ defined for $z = \alpha + i\beta \in \mathbb{C}$ as

$$W(\alpha + i\beta) = e^{i\alpha\beta/2}e^{i\alpha Q}e^{i\beta P}.$$

One can check that $W(z)$ is unitary for any $z \in \mathbb{C}$ and that

$$W(z)W(z') = e^{i\mathrm{Im}\langle z,z'\rangle}W(z+z'), \qquad z,z' \in \mathbb{C} \tag{2}$$

where $\langle z, z'\rangle = \bar{z}z'$ is the Hermitian scalar product of $\mathbb{C}$ (a one dimesional complex Hilbert space). All we are going to say generalises easily to $(W(z))_{z\in K}$ for finite dimensional Hilbert spaces K and strongly continuous unitary operators $(W(z))_{z\in K}$ such that (2) is satisfied.

Remark that $\omega(z,z') = \mathrm{Im}\langle z,z'\rangle$ is antisymmetric i.e. $\omega(z,z') = -\omega(z',z)$ and that $\omega(z,z') = 0$ for all z implies $z' = 0$ (i.e. ω is non-degenerate).

Let $\tilde{W}(z,\lambda) = e^{i\lambda}W(z)$ for $\lambda \in \mathbb{R}$ then

$$\tilde{W}(z,\lambda)\tilde{W}(z',\lambda') = W(z+z', \lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle),$$

which means that the $(\tilde{W}(z,\lambda))_{z,\lambda}$ give a unitary representation of the *Heisenberg group* $\mathbb{H} \approx \mathbb{C}\times\mathbb{R}$ with composition $(z,\lambda)(z',\lambda') = (z+z', \lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle)$. It a non-commutative group since ω is not symmetric.

Theorem 1. *(Von Neumann) Regular irreducible representations of the (finite dimensional) Weyl relations are all unitarily equivalent, i.e there is only one up to isomorphism.*

Remark 2. This theorem is fundamental because allows to use the most convenient representation to study the QM of finitely many quantum degrees of freedom (given by Weyl relations). Historically QM was developed independently by Schrödinger and Heisenberg (with Born and Jordan), then Dirac ('20) showed (formally) that the two approaches were unitarily equivalent. And later on Von Neumann ('30-'40) closed the matter by showing that there are no other possible representations. The theorem is false in infinite dimensions (and for physically motivated reasons).

Proof. (one dimensional case) Let us introduce the operator

$$P := \int_{\mathbb{R}^2} \mathrm{d}\alpha\,\mathrm{d}\beta\, e^{-(|\alpha|^2+|\beta|^2)/4}e^{i\alpha\beta/2}e^{i\alpha Q}e^{i\beta P} = \int_{\mathbb{C}} e^{-|z|^2/4}W(z)\mathrm{d}z\mathrm{d}\bar{z}$$

which is well defined as a strong integral, i.e when computed on vectors $\psi \in H$ (regularity is needed here, at least). We can check that $P \neq 0$ by observing that

$$W(-w)W(z)W(w) = e^{i\mathrm{Im}\langle z,w\rangle}W(-w)W(z+w) = e^{i\mathrm{Im}\langle z,w\rangle}e^{i\mathrm{Im}\langle -w,z+w\rangle}W(z) = e^{i2\mathrm{Im}\langle z,w\rangle}W(z)$$

and looking at

$$W(-w)PW(-w) = \int_{\mathbb{C}} e^{-|z|^2/4}W(-w)W(z)W(-w)\mathrm{d}z\mathrm{d}\bar{z} = \int_{\mathbb{C}} e^{-|z|^2/4}e^{i2\mathrm{Im}\langle z,w\rangle}W(z)\mathrm{d}z\mathrm{d}\bar{z}$$

Assume that $P = 0$, so we have $W(-w)PW(-w) = 0$ and for any vector $\psi \in H$ we will have for any $w \in \mathbb{C}$

$$0 = \int_{\mathbb{C}} e^{-|z|^2/4}e^{i2\mathrm{Im}\langle z,w\rangle}\langle\psi, W(z)\psi\rangle\mathrm{d}z\mathrm{d}\bar{z}$$

by Fourier transform with respect to both real and imaginary part of w we deduce that $e^{-|z|^2/4}\langle\psi, W(z)\psi\rangle = 0$ for almost all $z \in \mathbb{C}$ and by continuity of this function we have that $\langle\psi, W(z)\psi\rangle = 0$ for all z, and ψ but this is in contradiction with $W(0) = 1$. So $P \neq 0$.

With a tedious but elementary computartion with Fubini theorem and Gaussian integrals one can check that (exercise)

$$PW(w)P = e^{-|w|^2/4}P, \qquad w \in \mathbb{C}$$

so in particular this says that $P^2 = P$ and since is clear by definition that $P^* = P$ we have that that P is a non-trivial projection (it cannot be $P = 1$). So let ψ_0 be a unit vector in $\mathrm{Im}(P)$ so that $P\psi_0 = \psi_0$.

By irreducibility the linear space $\mathcal{D} := \mathrm{span}\{W(z)\psi_0 : z \in \mathbb{C}\}$ is dense in H since any element of the C^*-algebra generated by $(W(z))_{z\in\mathbb{C}}$ is a linear combination of $W(z)$s. We have also that ψ_0 is the only eigenvector of P since if φ is another one orthogonal to ψ_0 we have

$$\langle \varphi, W(z)\psi_0\rangle = \langle P\varphi, W(z)P\psi_0\rangle = \langle \varphi, PW(z)P\psi_0\rangle = e^{-|w|^2/4}\langle \varphi, \psi_0\rangle$$

so we learn that $\langle \varphi, W(z)\psi_0\rangle = 0$ for all z but then $\langle \varphi, \psi\rangle = 0$ for all $\psi \in \mathcal{D}$ and this implies that $\varphi = 0$. We learned also that there is a state ω such that

$$\omega_0(W(z)) = \langle \psi_0, W(z)\psi_0\rangle = e^{-|w|^2/4}.$$

Therefore we conclude that on any irreducible Weyl system there is a state ω such that

$$\omega_0(W(z)) = e^{-|w|^2/4}$$

(this relation define ω_0 on the full C^*-algebra, because any element can be approx. by linear comb of Ws).

Now if $(H, (W(z))_{z\in\mathbb{C}})$ and $(H', (W'(z))_{z\in\mathbb{C}})$ are two irreducible regular representations of the Weyl algebra we can construct a unitary operator $U: H \to H'$ by extending by linearity the equality

$$UW(z)\psi_0 = W'(z)\psi_0'$$

to the full $\mathcal{D}$ and observe that U is unitary since

$$\langle UW(z)\psi_0, UW(w)\psi_0\rangle = \langle W'(z)\psi_0', W'(w)\psi_0'\rangle = \langle \psi_0', PW'(-z)W'(w)P\psi_0'\rangle$$

$$= e^{-i\mathrm{Im}\langle z,w\rangle}\langle \psi_0', PW'(w-z)P\psi_0'\rangle = e^{-i\mathrm{Im}\langle z,w\rangle}e^{-|w-z|^2/4} = \langle W(z)\psi_0, W(w)\psi_0\rangle$$

therefore is bounded and can be extended to a unitary operator on the whole H. This show that the two representations of the Weyl relations are unitarily equivalent. □

The regular state ω_0 such that

$$\omega_0(W(z)) = e^{-|z|^2/4}$$

is called Fock vaccuum or vaccuum state for the Weyl representation. As a corollary, if a regular state is not given by ω_0 then it is not pure because there is only this pure regular state.

For example the state corresponding to $\omega_\alpha(W(z)) = e^{-\left(\frac{1}{4}+\alpha\right)|z|^2}$ for $\alpha > 0$ is not pure and can be decomposed into pure state and corresponding irreducible components.

Since the representation of the Weyl relation is essentially unique we could think to use the one we like (or the one more convenient).

One of them is the Schrödinger representation which is given on $H = L^2(\mathbb{R})$ by taking

$$U(t)f(x) = e^{itx}f(x), \qquad V(s)f(x) = f(x-s), \qquad f \in H, t, s \in \mathbb{R}.$$

Is this irreducible? If it is not irreducible then there exists two unit vectors $f, g \in L^2(\mathbb{R})$ such that for all t, $s \in \mathbb{R}$

$$0 = \langle f, U(t)V(s)g\rangle = \int_{\mathbb{R}} \bar{f}(x)e^{itx}g(x-s)\mathrm{d}x.$$

But then if this is true for any t we have that (by Fourier transform) $|\bar{f}(x)g(x-s)| = 0$ for almost every s and x, by integrating against a compactly supported smooth function $\varphi(s)$ we obtain that

$$0 = |\bar{f}(x)| \int_{\mathbb{R}} |g(x-s)|\varphi(s)\mathrm{d}s$$

and now since $x \mapsto \int_{\mathbb{R}} |g(x-s)|\varphi(s)\mathrm{d}s$ is smooth are arbitrarily supported somewhere we need to have that $|\bar{f}(x)| = 0$ for a.e. $x \in \mathbb{R}$. So this contradicts that g, f are unit vectors and proves that the Schrödinger representation is irreducible therefore there must exist a vector $\psi_0 \in L^2(\mathbb{R})$ such that

$$\langle \psi_0, e^{its/2}U(t)V(s)\psi_0\rangle = \exp\left(-\frac{1}{2}(s^2+t^2)\right).$$
