Lecture 14 – June 3rd 2020 – 8:15 via Zoom – M. Gubinelli

Canonical commutation relations / Quantum particle kinematics.

Recall. Canonical commutation relations (Weyl form)

$$U(t)V(s) = V(s)U(t)\exp(ist), \qquad s,t \in \mathbb{R}. \tag{1}$$

Where U, V are two unitary representations of the additive group of the reals, i.e.

$$U(t)U(s) = U(t+s), \qquad U(t)^* = U(-t),$$

and similarly for V. We can form the Weyl operators $(W(z))_{z\in\mathbb{C}}$ defined for $z = \alpha + i\beta \in \mathbb{C}$ as

$$W(\alpha + i\beta) = e^{-i\alpha\beta/2} e^{i\alpha Q} e^{i\beta P}.$$

One can check that $W(z)$ is unitary for any $z \in \mathbb{C}$ and that

$$W(z)W(z') = e^{i\mathrm{Im}\langle z,z'\rangle} W(z+z'), \qquad z, z' \in \mathbb{C} \tag{2}$$

where $\langle z, z'\rangle = \bar{z}z'$ is the Hermitian scalar product of $\mathbb{C}$ (a one dimesional complex Hilbert space).

Remark 1. All we are saying generalises easily to $(W(z))_{z\in K}$ for finite dimensional Hilbert spaces K and strongly continuous unitary operators $(W(z))_{z\in K}$ such that (2) is satisfied.

► **Schrödinger representation.** This is given on $H = L^2(\mathbb{R})$ by taking

$$U(t)f(x) = e^{itx}f(x), \qquad V(s)f(x) = f(x-s), \qquad f \in H, t, s \in \mathbb{R}.$$

Is this irreducible? If it is not irreducible then there exists two unit vectors $f, g \in L^2(\mathbb{R})$ such that for all t, $s \in \mathbb{R}$

$$0 = \langle f, U(t)V(s)g\rangle = \int_{\mathbb{R}} \bar{f}(x) e^{itx} g(x-s)\mathrm{d}x.$$

But then if this is true for any t we have that (by Fourier transform) $|\bar{f}(x)g(x-s)| = 0$ for almost every s and x, by squaring and integrating in x, s we have

$$0 = \int \mathrm{d}x \int \mathrm{d}s |\bar{f}(x)g(x-s)|^2 = \|f\|_{L^2}^2 \|g\|_{L^2}^2 = 1$$

so we have a contradiction and this proves that the Schrödinger representation is irreducible.

Therefore there must exist a vector $\psi_0 \in L^2(\mathbb{R})$ such that

$$\langle \psi_0, e^{-its/2} U(t)V(s)\psi_0\rangle = \exp\left(-\frac{1}{4}(s^2+t^2)\right), \qquad s, t \in \mathbb{R}$$

and by taking $s = 0$ we have

$$\int |\psi_0(x)|^2 e^{itx} \mathrm{d}x = \exp\left(-\frac{t^2}{4}\right)$$

which means that $|\psi_0(x)|^2$ is a Gaussian function (actually the density of a $\mathcal{N}(0,1/2)$ random variable), namely

$$|\psi_0(x)|^2 = \frac{1}{(\pi)^{1/2}} e^{-x^2}$$

this determines ψ_0 up to a phase factor: $\psi_0(x) = e^{if(x)} \frac{1}{(\pi)^{1/4}} e^{-x^2/2}$. However

$$\exp\left(-\frac{s^2+t^2}{4}\right) = \langle \psi_0, e^{-its/2} U(t) V(s) \psi_0 \rangle = e^{-its/2} \int \mathrm{d}x e^{itx} e^{-if(x)} \frac{1}{(\pi)^{1/4}} e^{-x^2/2} e^{if(x-s)} \frac{1}{(\pi)^{1/4}} e^{-(x-s)^2/2}$$

$$= \frac{e^{-its/2}}{(\pi)^{1/2}} \int \mathrm{d}x e^{it(x+s/2)} e^{-if(x+s/2)} e^{-(x+s/2)^2/2} e^{if(x-s/2)} e^{-(x-s/2)^2/2}$$

$$= \frac{e^{-s^2/4}}{(\pi)^{1/2}} \int \mathrm{d}x e^{-x^2} e^{itx} e^{i(f(x-s/2)-f(x+s/2))}$$

so we have

$$\frac{1}{(\pi)^{1/2}} \int \mathrm{d}x e^{itx} e^{i(f(x-s/2)-f(x+s/2))} e^{-x^2} = \exp\left(-\frac{t^2}{4}\right)$$

Now is better because this is saying that the function

$$\frac{1}{(\pi)^{1/2}} e^{i(f(x-s/2)-f(x+s/2))} e^{-x^2}$$

is the density of a Gaussian $\mathcal{N}(0,1/2)$ so it is equal to $\frac{1}{(\pi)^{1/2}} e^{-x^2}$ and we conclude that $f=0$, so we have proven that, in the Schrödinger representation we have

$$\psi_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}.$$

► **Gaussian representation.** We can introduce the unitary transformation (ground state transformation)

$$J: L^2(\mathbb{R}) \to L^2(\gamma)$$

where γ is the Gaussian measure with mean zero and variance $1/2$ by letting

$$(J\psi)(x) = \psi(x) / \psi_0(x), \qquad x \in \mathbb{R}.$$

Then we have the images U', V' of the Weyl pair U, V given by (for $f \in L^2(\gamma)$)

$$U'(t) f(x) = (JU(t)J^{-1}f)(x) = \psi_0(x)^{-1} U(t)(\psi_0 f)(x) = e^{itx} f(x)$$

$$V'(s) f(x) = (JV(s)J^{-1}f)(x) = \psi_0(x)^{-1} V(s)(\psi_0 f)(x) = \psi_0(x)^{-1} \psi_0(x-s) f(x-s)$$

$$= e^{xs-s^2/2} f(x-s)$$

One can check directly that this gives indeed a strongly continuous representation of the Weyl relation on $L^2(\gamma)$. This is called the Gaussian representation and is useful because there is a nice basis for $L^2(\gamma)$ given by polynomial functions, the Hermite basis $(h_n(x))_{n\geqslant 0}$ (indeed note that polynomials are in $L^2(\gamma)$ and that one can perform a Gram-Schmidt ortogonalisation procedure of the family $(x^n)_{n\geqslant 0}$ which is a separating family for $L^2(\gamma)$ by Stone-Weierstrass) and every $h_n(x)$ has monomial of highest degree n.

▸ **Reducible (regular) representations of Weyl relations.**

Assume now that $(W(z))_{z\in\mathbb{C}}$ does not act irreducibly on H then the range of P is not one dimensional. However in general we have that for any $\psi,\varphi\in H$

$$\langle W(z)P\varphi, W(z')P\psi\rangle = \langle\varphi, PW(z)^*W(z')P\psi\rangle = f(z,z')\langle P\varphi, P\psi\rangle$$

where we used that there exists a function f such that $f(z,z')P = PW(z)^*W(z')P$ and that does not depend on the specific representation. This means that I can compute it in any representation, in particular if we denote $\psi_0^\sharp$ the vacuum vector of the Schrödinger representation and by $(W^\sharp(z))_{z\in\mathbb{C}}$ the Weyl operators in the Schrödinger representation we have $\langle\psi_0^\sharp, P^\sharp W^\sharp(z)^* W^\sharp(z')P^\sharp\psi_0^\sharp\rangle_{L^2(\mathbb{R})} = f(z,z')$ and

$$\langle W(z)P\varphi, W(z')P\psi\rangle_H = \langle W^\sharp(z)P^\sharp\psi_0^\sharp, W^\sharp(z')P^\sharp\psi_0^\sharp\rangle_{L^2(\mathbb{R})}\langle P\varphi, P\psi\rangle_{\mathrm{Im}(P)} \tag{3}$$

therefore we can introduce a unitary operator $J: L^2(\mathbb{R})\otimes \mathrm{Im}(P)\to H$ defined by

$$J(W^\sharp(z)\psi_0^\sharp\otimes P\varphi) = W(z)P\varphi.$$

Remark 2. Let us recall that if K_1, K_2 are two Hilbert spaces there is a canonical notion of product of them, which is the Hilbert space $K_1\otimes K_2$ obtained by completing the span of all the monomials of the form $\{v\otimes w: v\in K_1, w\in K_2\}$ with respect to the Hermitian scalar product define on monomials by

$$\langle v_1\otimes w_1, v_2\otimes w_2\rangle_{K_1\otimes K_2} := \langle v_1, v_2\rangle_{K_1}\langle w_1, w_2\rangle_{K_2},$$

and extended by linearity (one has to check that this is a positive definite quantity, but for general results the product of positive definite kernels is a positive definite kernel).

Since $\{W^\sharp(z)\psi_0^\sharp\}_{z\in\mathbb{C}}$ span a dense subset of $L^2(\mathbb{R})$ (by irreducibility of the Schrödinger rep.) and $\{P\varphi\}_{\varphi\in H} = \mathrm{Im}(P)$ as Hilbert space, then J is well defined on all $L^2(\mathbb{R})\otimes\mathrm{Im}(P)$ and by construction it is isometric on H by (3). It remains to check that it is surjective. Let $\varphi\notin\mathrm{Im}(J)$ then we must have for any vector of the form $W(z)PW(-z)\psi$ since these are surely in the image of J, so for any $z\in\mathbb{C}$ and $\psi\in H$ we have

$$0 = \langle\varphi, J(W^\sharp(z)\psi_0^\sharp\otimes PW(-z)\psi)\rangle = \langle\varphi, W(z)PW(-z)\psi\rangle$$

recalling the definition of P we have

$$0 = \int_{\mathbb{C}} e^{-|w|^2/4}\langle\varphi, W(z)W(w)\,W(-z)\psi\rangle \mathrm{d}w\mathrm{d}\bar{w} = \int_{\mathbb{C}} e^{-|w|^2/4}e^{-2\mathrm{Im}\langle z,w\rangle}\langle\varphi, W(w)\psi\rangle\mathrm{d}w\mathrm{d}\bar{w}$$

since this has to be zero for any $z\in\mathbb{C}$ we deduce by Fourier transform that $\langle\varphi, W(w)\psi\rangle = 0$ for a.e. w but is also continuous in w so it is zero for all $w\in\mathbb{C}$ and then also for any $\psi\in H$. By taking $w=0$ and $\psi=\varphi$ we deduce that $\|\varphi\| = 0$ so $\varphi = 0$. In this way we proved that J is surjective and therefore that it is unitary.

Corollary 3. *Any regular representation $((W(z))_{z\in\mathbb{C}}, H)$ of the Weyl relations is unitarily equivalent to the representation $((W^\sharp(z))_{z\in\mathbb{C}}, L^2(\mathbb{R})\otimes K)$ where $K = PH$ and $W^\sharp(z)$ acts trivially on K and as the Schrödinger representation on $L^2(\mathbb{R})$, i.e.*

$$W^\sharp(z)(\psi^\sharp\otimes\psi^\natural) = (W^\sharp(z)\psi^\sharp)\otimes\psi^\natural,\qquad z\in\mathbb{C}, \psi^\sharp\in L^2(\mathbb{R}), \psi^\natural\in K.$$

Next time I will continue to discuss the Weyl relation: other ways to consrtuct reducible representations and Fock representation, dynamics: free particle and harmonic oscillator.