Lecture 15 – 2020.6.9 – 14:15 via Zoom – M. Gubinelli

Canonical commutation relations / Quantum particle kinematics.

We know that the only regular irreducible representation on a Hilbert space H of the Weyl relations is given by a state such that

$$\omega(W(z)) = e^{-|z|^2/4}.$$

This state corresponds to a cyclic vector $\psi_0 \in H$ by means of the relation $\omega(a) = \langle \psi_0, a\psi_0 \rangle$ which defines a state on $\mathscr{L}(H)$, we have also that the weak closure of the Weyl algebra $(W(z))_{z\in\mathbb{C}}$ is the whole $\mathscr{L}(H)$.

Moreover any state with the same expectation of the Weyl operators give rise to a representation (via GNS construction) which is unitarily equivalent with the Schrödinger representation, in particular it is irreducible.

How do reducible representations looks like. I want to give an example. The easiest way to come up with a reducible representation is to that two copies $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$ of the Schrödinger representation and define Weyl operators

$$(\tilde{W}(s+it)f)(x_1, x_2) = (e^{its/2}\tilde{U}(s)\tilde{V}(t)f)(x_1, x_2)$$

$$= e^{its/2}e^{is(ax_1+bx_2)}f(x_1 - at, x_2 + bt) = e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt)$$

where (U_1, V_1) and (U_2, V_2) are Weyl pairs acting independenlty on the two factors of $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, so they commute among them. We can check that they satisfy the Weyl relations

$$\tilde{W}(s+it)\tilde{W}(s'+it') = e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt)e^{it's'/2}U_1(as')U_2(bs')V_1(at')V_2(-bt')$$

$$= e^{its/2}e^{it's'/2}(U_1(as')U_2(bs')V_1(at')V_2(-bt'))(U_1(as)U_2(bs)V_1(at)V_2(-bt))e^{-i(-bt)(bs')-i(at)(as')+i(bs)(-bt')+i(as)(at')}$$

$$= e^{i(b^2s't - a^2s't - b^2st' + a^2st')}\tilde{W}(s+it)\tilde{W}(s'+it') = e^{-i(b^2-a^2)\operatorname{Im}[(\overline{s+it})(s'+it')]}\tilde{W}(s'+it')\tilde{W}(s+it)$$

iff $a^2 - b^2 = 1$. This also implies that the operators $\tilde{W}$ are unitary, indeed

$$(e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt))^* = e^{-its/2}V_2(bt)V_1(-at)U_2(-bs)U_1(-as)$$

$$= e^{-its/2}e^{i(-as(-at))}e^{i(-bs(bt))}U_1(-as)U_2(-bs)V_1(-at)V_2(bt)$$

$$= e^{-its/2}e^{i((a^2-b^2)st)}U_1(-as)U_2(-bs)V_1(-at)V_2(bt) = \tilde{W}(-s-it).$$

In this way we can construct a family of Weyl pairs. Let $\Psi_0 = \psi_0 \otimes \psi_0$ the tensor product of the two vacuum states, then

$$\langle \psi_0 \otimes \psi_0, \tilde{W}(s+it)(\psi_0 \otimes \psi_0)\rangle_{L^2(\mathbb{R}^2)} = e^{i(a^2-b^2)ts/2}\langle \psi_0, U_1(as)V_1(at)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, U_2(bs)V_2(-bt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= \langle \psi_0, e^{ia^2ts/2}U_1(as)V_1(at)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, e^{-ib^2ts/2}U_2(bs)V_2(-bt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= \langle \psi_0, W(as+iat)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, W(bs-ibt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= e^{-|as+ait|^2/4}e^{-|-bt+bis|^2/4} = e^{-(a^2+b^2)|s+it|^2/4} = e^{-(1+2b^2)|s+it|^2/4}$$

Theorem 1. *For any $Q \geqslant 1/2$ there exists a state ω_Q on the Weyl algebra such that*

$$\omega_Q(W(z)) = e^{-Q|z|^2/2}.$$

Moreover we know that for $Q = 1/2$ is pure (because it corresponds to the Schrödinger model) and for $Q > 1/2$ it is not.

Let us show concretely that the representation given by $\tilde{W}$ on $L^2(\mathbb{R}^2)$ is not irreducible. Consider the operators

$$(W^\sharp(s+it)f)(x_1,x_2) = e^{its/2} U_1(bs) U_2(as) V_1(-bt) V_2(at) = W_1(bs-ibt) W_2(as+iat)$$

and note that

$$\tilde{W}(s'+it') W^\sharp(s+it) = W_1(as+iat) W_2(bs-ibt) W_1(bs-ibt) W_2(as+iat)$$

$$= \underbrace{e^{i\mathrm{Im}\langle as+iat, bs-ibt\rangle} e^{i\mathrm{Im}\langle bs-ibt, as+iat\rangle}}_{=1} W_1(bs-ibt) W_2(as+iat) W_1(as+iat) W_2(bs-ibt)$$

$$= W^\sharp(s+it) \tilde{W}(s'+it')$$

so the two families commute. In particular the Stone–von Neumann projector $P^\sharp$ associated to the Weyl system $W^\sharp$ satisfy

$$P^\sharp \tilde{W}(z) = \tilde{W}(z) P^\sharp$$

and therefore $(W^\sharp(z))_{z\in\mathbb{C}}$ is not an irreducible representation since $P^\sharp$ is a non-trivial self-adjoint operator. Moreover if $\psi_0^\sharp \in L^2(\mathbb{R}^2)$ is a unit vector such that $P^\sharp \psi_0^\sharp = \psi_0^\sharp$ then the space $K = \overline{\{W^\sharp(z)\psi_0^\sharp : z \in \mathbb{C}\}}^{L^2(\mathbb{R}^2)}$ is invariant under the action of $\tilde{W}(z)$ and we have that $\{\tilde{W}(z)K : z \in \mathbb{C}\}$ is dense in $L^2(\mathbb{R}^2)$.

Question 1. It is a fact that there not exists states on the Weyl algebra for which

$$\omega_Q(W(z)) = e^{-Q|z|^2/2},$$

with $Q < 1/2$. How to prove it? (one possible attempt is to prove that ω_Q is dominated by $\omega_{1/2}$, in the sense that $\omega_{1/2}$ could be written as a linear combination of ω_Q and other states which is impossible by irreducibility, maybe use product of two representations).

Dynamics on a canonical pair

Note that if $(W(z))_{z\in\mathbb{C}}$ is an irreducible Weyl system on some Hilbert space H then also

$$(\tilde{W}_t(z) = W(e^{it}z))_{z\in\mathbb{C}}$$

is a Weyl system for any $t \in \mathbb{R}$. Then it must be that there exists a unitary operator U_t such that

$$U_t \tilde{W}_t(z) U_t^* = W(z), \qquad t \in \mathbb{R}, z \in \mathbb{C}.$$

Moreover we can define an automorphism of the Weyl algebra by letting $\alpha_t(W(z)) = W(e^{it}z)$ (i.e. a map of the Weyl algebra in itself which respects the $*$-operation and the algebraic relations in the C^*-algebra, and as a consequence is an isometry). This is an example of *dynamics*, i.e. the introduction of a time evolution in our description of a physical system.

Next time I will provide some more detail on which kind of dynamics this transformation describes.

Let us obseve that $\alpha_{2\pi}(W(z)) = W(z)$ so $\alpha_{2\pi} = \mathrm{id}$. So the dynamics is periodic of period 2π, we will see that it corresponds to the quantum motion of an harmonic oscillator.