Lecture 16 – 2020.6.10 – 8:15 via Zoom – M. Gubinelli

Dynamics on a canonical pair

So far we described the kinematics, that is the structure of the space of observables which holds at a specific time, because we imagine to perform a measurement described by $\mathscr{A}$ on a state ω.

In order to do predictions one has to correlate the measurements on the same system at different times: we have a model which given information from the past allows us to predict the future. (that's one of the basic goals of physics).

The time and dynamics enters into the model via a group $(\alpha_t)_t$ of $(*$-)automorphisms of $\mathscr{A}$, which have the following meaning $\omega(\alpha_t(a))$ is the measurement of the observable a at the time t. $\alpha_0 = \mathrm{id}$. $\alpha_{t+s} = \alpha_t \circ \alpha_s$, i.e. is a representation of the additive group of $\mathbb{R}$ onto automorphisms of the C^*-algebra $\mathscr{A}$.

We can let α act on the linear functional by duality: $(\alpha_t^* \varphi)(a) := \varphi(\alpha_t(a))$ and then this gives a group of linear transformations on linear functionals on $\mathscr{A}$ and is easy to see that it preserves the states of $\mathscr{A}$.

Suppose that $\alpha_t^* \omega$ is not pure, then it can be decomposed into two states $\alpha_t^* \omega = \lambda \omega_1 + (1 - \lambda) \omega_2$ but then $\omega = \alpha_{-t}^* \alpha_t^* \omega = \lambda \alpha_{-t}^* \omega_1 + (1 - \lambda) \alpha_{-t}^* \omega_2$ so ω is not pure either. Therefore the dynamics preseves pure states.

Moreover one assume that there is some continuity in the time evolution, namely that $t \mapsto \omega(\alpha_t(a))$ is continuous in t for all a, ω.

Additionally we make the crucial assumptions that the orbit of a given pure state lies within its folium. Therefore if ω is pure and give rise to a GNS representation $(H_\omega, \pi_\omega, \Omega_\omega)$ then since $\alpha_t^* \omega$ it is in the same folium and its pure by the consideration before we can represent it by a vector state $\psi(t) \in H_\omega$

$$\alpha_t^* \omega(a) = \langle \psi(t), \pi_\omega(a) \psi(t) \rangle$$

but this can also be written as

$$\omega(\alpha_t(a)) = \langle \Omega_\omega, \pi_\omega(\alpha_t(a)) \Omega_\omega \rangle$$

therefore one can introduce a unitary operator U_t such that

$$U_t \pi_\omega(\alpha_t(a)) \Omega_\omega = \pi_\omega(a) \psi(t)$$

and in particular $U_t \Omega_\omega = \psi(t)$ and $\pi_\omega(\alpha_t(a)) = U_t^{-1} \pi(a) U_t$. One can check that U_t is indeed unitary

$$\langle U_t \pi_\omega(\alpha_t(a)) \Omega_\omega, U_t \pi_\omega(\alpha_t(b)) \Omega_\omega \rangle = \langle \pi_\omega(a) \psi(t), \pi_\omega(b) \psi(t) \rangle = \langle \psi(t), \pi_\omega(a^* b) \psi(t) \rangle$$

$$= \alpha_t^* \omega(a^* b) = \langle \Omega_\omega, \pi_\omega(\alpha_t(a^* b)) \Omega_\omega \rangle = \langle \pi_\omega(\alpha_t(a)) \Omega_\omega, \pi_\omega(\alpha_t(b)) \Omega_\omega \rangle$$

and similarly one can check that $U_{t+s} = U_t U_s$ and that $U_t \psi(s) = \psi(t+s)$. Moreover one talks about Schrödinger picture when looking at means of the form

$$\langle \psi(t), \pi_\omega(a) \psi(t) \rangle$$

where the state depends on time and the observables not and of Heisenberg's picture when looking at

$$\langle \Omega_\omega, \pi_\omega(\alpha_t(a)) \Omega_\omega \rangle$$

with a fixed state representing the initial condition and the observables which depends on time.

If I assume that $\alpha_t^*\rho$ is weak-* continuous for all ρ in the folium of ω then we have that

$$\omega(b^*\alpha_t(a)b)$$

is continuous in t for fixed $a, b \in \mathscr{A}$ and by polarization and it also means that

$$\omega(c^*\alpha_t(a)b) = \frac{1}{4}\sum_{k=0}^{3} i^k \omega((b+i^k c)^*\alpha_t(a)(b+i^k c))$$

since

$$i^k\omega((b+i^k c)^*\alpha_t(a)(b+i^k c)) = i^k\omega(b^*\alpha_t(a)b) + i^{2k}\omega(b^*\alpha_t(a)c) + \omega(c^*\alpha_t(a)b) + i^k\omega(c^*\alpha_t(a)c).$$

Therefore by taking $b = 1$ we have that $t \mapsto \omega(c^*\alpha_t(a))$ is continuous in t which implies that

$$\omega(c^*\alpha_t(a)b) = \langle \pi_\omega(c)\Omega_\omega, \pi_\omega(\alpha_t(a))\pi_\omega(b)\Omega_\omega\rangle$$

Remark 1. During the lecture I encountered a difficulty to deduce from this informations that U is a strongly continuous unitary group. This is crucial for the continuation of this discussion. I will write a note on how this work.