Lecture 17 – 2020.6.16 – 14:15 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Commutative setting: representation Q_0 of an abelian C^* algebra $\mathscr{A}$ on an Hilbert space $\mathscr{H}$.

$$\mathscr{A} = C_b^0(\mathbb{R}^n, \mathbb{C}), \qquad \mathscr{H} = L^2(\mathbb{R}^n, \mathbb{C})$$

$$a(x) \in \mathscr{A}, \qquad h(x) \in \mathscr{H}$$

$$Q_0(a)h = a(x)h(x)$$

$$Q_0(e^{iax})h = e^{iax}h(x)$$

Norm on $\mathscr{A}$ is the uniform norm on $C_b^0(\mathbb{R}^n, \mathbb{C})$. This representation is faithful $\ker(Q_0) = 0$.

Suppose that we have a cyclic vector $h_0 \in \mathscr{H}$.

$$\mathscr{H}_0 = \{Q_0(a)h_0, a \in \mathscr{A}\}, \qquad \overline{\mathscr{H}_0} = \mathscr{H}.$$

Theorem 1. *Under the hypothesis $\overline{\mathscr{H}_0} = \mathscr{H}$ the system $(\mathscr{H}, \mathscr{A}, Q_0)$ is isomorphic to $(L^2(X, \mathbb{C}, \mu), C_\infty^0(X, \mathbb{C}), m)$ where X is a locally compact Hausdorff space, μ is a measure on X and C_∞^0 is the set of continuous functions going to zero at infinity and m is the multiplication operator.*

Proof. By Gelfand–Naimark $\mathscr{A} \approx C_\infty^0(X, \mathbb{C})$ where X is the space of characters (i.e. pure, positive states on $\mathscr{A}$) equipped with the weak-* topology. □

Remark 2. In the case where $1 \in \mathscr{A}$ then X is compact, so $\mathscr{C}_\infty^0(X) = \mathscr{C}^0(X)$.

We can take $\mathscr{H} = \mathscr{H}^{\text{GNS}}$ where the state generating the GNS construction is $\omega^{h_0}(a) = \langle h_0, Q_0(a)h_0 \rangle$. Here ω^{h_0} is a positive functional on $\mathscr{A}$. ω^{h_0} is continuous wrt. the $\|\cdot\|_\infty$ norm where we identify $\mathscr{A} \approx C_\infty^0(X, \mathbb{C})$. So ω^{h_0} defines a measure on X since is in $(C_\infty^0(X, \mathbb{C}))^*$ (the dual space, i.e. the space of bounded measures). Moreover it is a non-negative measure. We call it μ and have that

$$\mathscr{H}^{\text{GNS}} \to L^2(X, \mu)$$

$$U(Q_0(a)h_0) = a(x) \in L^2(X, \mu)$$

This is an isomorphism where Q_0 corresponds to the multiplication m.

Let us note that we have that $\mathbb{R}^n \hookrightarrow X$ and actually X is a compactification of $\mathbb{R}^n$ which we are not able to work with explicitly.

Dynamics

$(\mathscr{H}, \tilde{\mathscr{A}}, \tilde{Q}_0)$ where $\tilde{\mathscr{A}}$ is a general C^*-algebra and $\tilde{Q}_0$ is a representation in $\mathscr{H}$.

Definition 3. *Let $(\alpha_t)_{t\in\mathbb{R}}$ a set of C^*-automorphisms of $\tilde{\mathscr{A}}$. We call α a regular dynamics, if*

i. $(\alpha_t)_{t\in\mathbb{R}}$ *is a group wrt. t, i.e.* $\alpha_0 = \mathrm{id}$ *and* $\alpha_t \circ \alpha_s = \alpha_{t+s}$ *for any* $t, s \in \mathbb{R}$

ii. the map $t \mapsto \alpha_t$ *is weakly continuous, i.e. for any state* ω *and for any* $a \in \tilde{\mathscr{A}}$ *the map* $t \mapsto \omega(\alpha_t(a))$ *is continuous.*

Define $\tilde{Q}_t(a) := \tilde{Q}(\alpha_t(a))$ for $a \in \tilde{\mathscr{A}}$

Definition 4. *The set* $\{U(t)\}_{t\in\mathbb{R}} \subset \mathscr{B}(\mathscr{H})$ *is a unitary group of strongly continuous operators, if* $U(t)U(s) = U(t+s)$ *and* $U(t)^* = U(-t)$ *and if the map* $t \mapsto U(t)$ *is weakly (and thus strongly) continuous.*

Theorem 5. *Assume that there exists a state* $\omega^{h_0}(\alpha_t(a)) = \omega^{h_0}(a)$ *for all* $t \in \mathbb{R}$ *and* $a \in \tilde{\mathscr{A}}$ *and* $(\alpha_t)_t$ *is a regular dynamics of* $\tilde{\mathscr{A}}$*, then if* $\mathscr{H}$ *is the GNS representation space associated with* ω^{h_0} *and* $h_0 \in \mathscr{H}$ *is the corresponding cyclic vector, then there exists a unitary strongly continuous group* $(U(t))_{t\in\mathbb{R}}$ *on* $\mathscr{H}$ *such that*

$$\tilde{Q}_t(\cdot) = U(t)\tilde{Q}_0(\cdot)U(-t)$$

and also $U(t)h_0 = h_0$.

Lemma 6. *Suppose that we have a contraction* $V(t)$*, i.e.* $\|V(t)h\| \leqslant \|h\|$*, such that* $V(0) = 1$ *and* $V(t)$ *is weakly continuous in t at zero, then it is strongly continuous at zero.*

Proof. We have

$$0 \leqslant \|V(t)h - h\|^2_{\mathscr{H}} = \|V(t)h\|^2_{\mathscr{H}} + \|h\|^2_{\mathscr{H}} - 2\mathrm{Re}\langle V(t)h, h\rangle_{\mathscr{H}} \leqslant 2\|h\|^2_{\mathscr{H}} - 2\mathrm{Re}\langle V(t)h, h\rangle_{\mathscr{H}}$$

so weak continuity at zero is enough for strong continuity at zero. □

Proof. (of the Theorem 5)

$$\mathscr{H}_0 = \{\widetilde{Q}_0(a)h_0 | a \in \tilde{\mathscr{A}}\}, \qquad \overline{\mathscr{H}_0} = \mathscr{H},$$

Let's define

$$U_0(t)(\widetilde{Q}_0(a)h_0) = \widetilde{Q}_t(a)h_0 = \widetilde{Q}_0(\alpha_t(a))h_0$$

We first prove that $U_0(t)$ is an isometry

$$\langle U_0(t)(\widetilde{Q}_0(a_1)h_0), U_0(t)(\widetilde{Q}_0(a_2)h_0)\rangle = \langle \widetilde{Q}_0(\alpha_t(a_1))h_0, \widetilde{Q}_0(\alpha_t(a_2))h_0\rangle$$

$$= \langle h_0, \widetilde{Q}_0(\alpha_t(a_1))^*\widetilde{Q}_0(\alpha_t(a_2))h_0\rangle = \langle h_0, \widetilde{Q}_0(\alpha_t(a_1^* a_2))h_0\rangle = \omega^{h_0}(\alpha_t(a_1^* a_2))$$

$$= \omega^{h_0}(a_1^* a_2) = \langle h_0, \widetilde{Q}_0(a_1^* a_2)h_0\rangle = \langle \widetilde{Q}_0(a_1)h_0 h_0, \widetilde{Q}_0(a_2)h_0\rangle$$

So $U_0(t)$ is an isometry on $\mathscr{H}_0$ so it is bounded on $\mathscr{H}_0$ and can be extended by continuity to $\overline{\mathscr{H}_0} = \mathscr{H}$. It remains to prove that it form a group. $\alpha_0 = 1 \Rightarrow U_0(0) = I_{\mathscr{H}}$ and

$$U_0(t)U_0(s)(\widetilde{Q}_0(a)h_0) = U_0(t)(\widetilde{Q}_0(\alpha_s(a))h_0) = (\widetilde{Q}_0(\alpha_t(\alpha_s(a)))h_0) = (\widetilde{Q}_0(\alpha_{t+s}(a))h_0) = U_0(t+s)(\widetilde{Q}_0(a)h_0)$$

so $U_0(t)U_0(s) = U_0(t+s)$ on $\mathcal{H}_0$ and therefore on all $\mathcal{H}$. It remains to prove that h_0 is invariant, but of course $U_0(t)h_0 = U_0(t)(\widetilde{Q}_0(1)h_0) = \widetilde{Q}_0(\alpha_t(1))h_0 = h_0$. We also have that it is weakly continuous

$$\langle (\widetilde{Q}_0(a)h_0), U_0(t)(\widetilde{Q}_0(b)h_0)\rangle = \langle h_0, \widetilde{Q}_0(a^*\alpha_t(b))h_0\rangle = \omega^{h_0}(a^*\alpha_t(b))$$

and $\omega^{h_0}(a^*\cdot)$ is a continuous functional on $\mathcal{A}$ and therefore $t\mapsto \omega^{h_0}(a^*\alpha_t(b))$ is continuous, which proves that $U_0(t)$ is weaky continuous in $\mathcal{H}_0$ and then strongly continuous and can be extended as a strongly continuous group in $\mathcal{H}$. Note finally that

$$\tilde{Q}_t(a)\tilde{Q}_0(b)h_0 = \tilde{Q}_t(a\alpha_{-t}(b))h_0 = U_0(t)(\widetilde{Q}_0(a\alpha_{-t}(b))h_0) = U_0(t)(\widetilde{Q}_0(a)\widetilde{Q}_0(\alpha_{-t}(b))h_0)$$

$$= U_0(t)\widetilde{Q}_0(a)U_0(-t)\widetilde{Q}_0(b)h_0$$

so this proves that $\tilde{Q}_t(a) = U_0(t)\widetilde{Q}_0(a)U_0(-t)$. □

Without the hypothesis that the state is invariant, then this construction is not true in general anymore. Take for example $\mathcal{A}$ commutative, i..e $C^0_\infty(\mathbb{R}^2)$ and consider an Hilbert space $L^2(\mathbb{R}^2,\mu)$ where

$$\mu(\mathrm{d}x) = e^{-x^2/2}\mathrm{d}x + \delta_0(\mathrm{d}x)$$

and the usual moltiplication and take $\alpha_t(f(x)) = f(x-t)$. But here there is no unitary group associated to α. Indeed take the state $\omega^\mu(a) = \int a(x)\mu(\mathrm{d}x)$. Consider the translated state $\omega^\mu(\alpha_t(\cdot))$, then GNS representation of it lives on $L^2(\mathbb{R}^n,\mu_t)$ where $\mu_t = T_t^*\mu$ the pull forward of μ by the translation operator. In order to have a unitary transformation we need that μ_t has to be absolutely continuous wrt. μ, but this is not the case.

In this lectures we will request always to have a unitary implementation of the dynamics $(\alpha_t)_{t\in\mathbb{R}}$ for $(\mathcal{H}, \mathcal{A}, Q_0)$, i.e. to have a strongly continuous group of unitary operators $(U(t))_{t\in\mathbb{R}}$ so that $Q_t(\cdot) = Q_0(\alpha_t(\cdot)) = U(t)Q_0(\cdot)U(-t)$.

Theorem 7. *Consider an Hilbert space $\mathcal{H}$, a strongly continuous unitary group $(U(t))_{t\in\mathbb{R}}$ on $\mathcal{H}$, then there exists a unique C^*-representation X of $C^0_b(\mathbb{R},\mathbb{C})$ on $\mathcal{H}$ such that*

i. $X(e^{it\cdot}) = U(t)$

ii. *If $f_n\to f$ pointwise and $\sup_n\|f_n\| < \infty$ then $X(f_n)\to X(f)$ weakly.*

This was proven in one of the last lectures.

Definition 8. $\{K(t)\}_{t\in\mathbb{R}_+}\subseteq \mathcal{B}(\mathcal{H})$. *We say that $K(t)$ is a strongly continuous semigroup of self-adjoint contractions if*

i. $K(0) = 1$, $K(t)K(s) = K(t+s)$, *for* $t,s\geqslant 0$.

ii. $K(t) = K(t)^*$,

iii. $t\mapsto K(t)$ *is strongly continuous*

iv. $\|K(t)h\|\leqslant\|h\|, t\geqslant 0$.

Next lecture we will prove the following theorem:

Theorem 9. *Assume that K is a strongly continuous semigroup of self-adjoint contractions then there exists a unique representation X of $C_b^0(\mathbb{R}_+)$ on $\mathcal{H}$ such that*

i. $X(e^{-t\cdot}) = K(t)$

ii. If $f_n \to f$ pointwise and $\sup_n \|f_n\| < \infty$ *then $X(f_n) \to X(f)$ weakly.*