Lecture 18 – 2020.6.17 – 8:30 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Definition 1. $\{K(t)\}_{t\in\mathbb{R}_+}\subseteq\mathscr{B}(\mathscr{H})$. *We say that* $K(t)$ *is a strongly continuous semigroup of self-adjoint contractions if*

i. $K(t)=K(t)^*$, $K(0)=1$, $K(t)K(s)=K(t+s)$, *for* $t,s\geqslant 0$.

ii. $t\mapsto K(t)$ *is strongly continuous*

iii. $\|K(t)h\|\leqslant\|h\|, t\geqslant 0$.

We want to prove now that

Theorem 2. *Assume that* K *is a strongly continuous semigroup of self-adjoint contractions then there exists a unique* $*$*-representation* X *of* $C_b^0(\mathbb{R}_+,\mathbb{C})$ *on* $\mathscr{H}$ *such that*

i. $X(e^{-t\cdot})=K(t)$

ii. *If* $f_n\to f$ *pointwise and* $\sup_n\|f_n\|<\infty$ *then* $X(f_n)\to X(f)$ *weakly.*

Definition 3. *If* $G:\mathbb{R}\to\mathbb{C}$ *we call* G *positive definite if for any* $\lambda_1,\ldots,\lambda_k\in\mathbb{C}$ *and* $t_1,\ldots,t_k\in\mathbb{R}$ *we have*

$$\sum_{i,j=1}^k \lambda_i\bar{\lambda}_j G(t_i-t_j)\geqslant 0$$

Definition 4. *We say that* $F:\mathbb{R}_+\to\mathbb{C}$ *is totally monotone if for any* $\lambda_1,\ldots,\lambda_k\in\mathbb{C}$ *and* $t_1,\ldots,t_k\in\mathbb{R}_+$ *we have*

$$\sum_{i,j=1}^k \lambda_i\bar{\lambda}_j F(t_i+t_j)\geqslant 0.$$

Take U a unitary group on $\mathscr{H}$. For any $h\in\mathscr{H}$ we define $F_U(t,h)=\langle U(t)h,h\rangle$. If K is a self-adjoint contration semigroup we define $F_K(t,h)=\langle K(t)h,h\rangle$.

Theorem 5. *Let* U *and* K *as before, then* F_U *is positive definite and* F_K *is totally monotone.*

Proof. Consider K, the case of U is similar. Take $\lambda_1,\ldots,\lambda_k\in\mathbb{C}$ and $t_1,\ldots,t_k\in\mathbb{R}_+$ and just compute

$$0\leqslant\left\langle\sum_i\lambda_i K(t_i)h,\sum_i\lambda_i K(t_i)h\right\rangle=\sum_{i,j=1}^k\lambda_i\bar{\lambda}_j F_K(t_i+t_j)$$

using the fact that K is self-adjoint and a semigroup. □

Theorem 6. *(Bochner) G is a continuous positive definite function iff there exists a bounded positive measure μ on $\mathbb{R}$ such that*

$$G(t) = \int_{\mathbb{R}} e^{itx} \mu(\mathrm{d}x).$$

Theorem 7. *(Bernstein) F is a bounded totally monotone function iff there exists a bounded positive measure μ on $\mathbb{R}_+$ and a constant $C \geqslant 0$ such that*

$$F(t) = C \int_{\mathbb{R}_+} e^{-tx} \mu(\mathrm{d}x).$$

Remark 8. These results can be generalises in a more abstract setting by replacing $\mathbb{R}$ and $\mathbb{R}_+$ with other topological groups/semigroups and exponentials with characters.

Lemma 9. *Assume that F is a bounded, totally monotone function, then*

a) For any $a > 0$, $-\Delta_a F$ is bounded totally monotone with $\Delta_a F(t) = F(t+a) - F(t)$.

Proof. $F \geqslant 0$, $a, t \geqslant 0$

$$\begin{pmatrix} F(2t) & F(t+a) \\ F(t+a) & F(2a) \end{pmatrix}$$

is positive definite, so its determinant is positive and

$$F(t+a) \leqslant \sqrt{F(2t)F(2a)}$$

Then (starting with $a = 0$)

$$F(t) \leqslant F(0)^{1/2} F(2t)^{1/2} \leqslant F(0)^{3/4} F(4t)^{1/4} \leqslant \cdots \leqslant F(0)^{(2^n-1)/2^n} F(2^n t)^{1/2^n} \leqslant F(0)^{(2^n-1)/2^n} C^{1/2^n}$$

and so we conclude that $F(t) \leqslant F(0)$. Take $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $t_1, \ldots, t_k \in \mathbb{R}_+$ and define

$$G(a) = \sum_{i,j}^{n} F(a + t_i + t_j) \lambda_i \bar{\lambda}_j$$

and consider other points $\sigma_1, \ldots, \sigma_n \in \mathbb{C}$ and $a_1, \ldots, a_n \in \mathbb{R}_+$ then

$$\sum_{i,j}^{k} G(a_i + a_j) \sigma_i \bar{\sigma}_j = \sum_{i,j}^{n} \sum_{r,s}^{k} F(a_i + a_j + t_r + t_s) \lambda_r \bar{\lambda}_s \sigma_i \bar{\sigma}_j \geqslant 0$$

using the fact that F is totally monotone. So G is also totally monotone and as a consequence $G(a) \leqslant G(0)$ and $G(0) - G(a) \geqslant 0$ or otherwise

$$\sum_{i,j}^{n} (-\Delta_a F(t_i + t_j)) \lambda_i \bar{\lambda}_j = \sum_{i,j}^{n} (F(t_i + t_j) - F(a + t_i + t_j)) \lambda_i \bar{\lambda}_j \geqslant 0$$

so $-\Delta_a F$ is bounded and totally monotone. □

Corollary 10. *If F is bounded and totally monotone, for any $a_1, \ldots, a_n \in \mathbb{R}_+$*

$$(-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F$$

is totally monotone and therefore $(-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F \geqslant 0$.

Theorem 11. *(Krein–Milman) Let X be a locally convex Hausdorff topological vector space and let $K \subseteq X$ be a compact convex subset, then the set $E(K)$ of extreme points of K is non-void and for any $y \in K$ there exists a probability measure ν^y on $E(K)$ such that*

$$y = \int_{E(K)} x \nu^y(\mathrm{d}x)$$

where the integral is understood in the weak sense, i.e. for any $\lambda \in X^*$ we have (Pettis integral)

$$\lambda(y) = \int_{E(K)} \lambda(x) \nu^y(\mathrm{d}x).$$

Recall that locally convex means that there is a base of the topology composed by convex sets. For example $\mathbb{R}^{(0,+\infty)}$ with the product topology is a locally convex and Hausdorff.

Proof. (of Bernstein theorem) We prove now that if F is bounded and totally monotone there exists a positive measure μ on $\mathbb{R}_+$ such that $F(t) = \int_{\mathbb{R}_+} e^{-tx} \mu(\mathrm{d}x)$. The rest of the claim is left as an exercise. Consider the space $\mathscr{C} \subseteq \mathbb{R}^{(0,\infty)}$ such that

$$\mathscr{C} = \{F \in \mathbb{R}^{(0,\infty)}, F \geqslant 0 \colon \text{for all } a_1, \ldots, a_n \in \mathbb{R}_+ \ (-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F \geqslant 0\}$$

Note that $\mathscr{C}$ is closed for the pointwise convergence and it is convex, but not compact. In particular this means that for $F \in \mathscr{C}$ we have $F(t_1) - F(t_2) \geqslant 0$ if $t_1 \leqslant t_2$ and we let $F(0+) = \lim_{t \downarrow 0} F(t)$ by monotone limit. In principle we could have $F(0+) = +\infty$. F is bounded iff $F(0+) < \infty$. Since $\Delta_a \Delta_a F \geqslant 0$ we have

$$\frac{1}{2} F(t) + \frac{1}{2} F(t + 2a) \geqslant F(t + a)$$

and this means that F is midpoint convext. On the other hand, for any $0 < c < d$ we have that $0 \leqslant F(d) \leqslant F(c)$ so F is bounded in $[c, d]$. It is left as an exercise to prove that if F is midpoint convex and bounded then F is continuous in (c, d) (Hint: show that $F \colon [-\delta, \delta] \to \mathbb{R}$ midpoint convex and if F has a discontinuity in 0 then it is unbounded). By this result, F is continuous on $\mathbb{R}_+$. Consider a subset $K \subseteq \mathscr{C}$ as follows $K = \{F \in \mathscr{C} \colon F(0+) = 1\}$. This is now a closed convex set and $K \subset [0,1]^{\mathbb{R}_+}$ which is a compact space (always wrt. to the pointwise convergence). By Krein–Milman this means that for any $y \in K$ we can write it as a convex combination of extreme points. What are these extreme points $E(K)$ of K? For any $F \in K$ we have that exists $a \in \mathbb{R}_+$ such that $F(a) > 0$ and $1 = F(0) > F(a) > 0$ unless $F = 1$ everywhere. In the second case $1 \in E(K)$ since it is the biggest element of K and therefore cannot be decomposed in a convex combination of other elements. In the other case

$$F(t) = \frac{F(t+a)}{F(a)} F(a) + \frac{-\Delta_a F(t)}{1 - F(a)} (1 - F(a))$$

so $F(t+a) / F(a) \in K \subseteq \mathscr{C}$ so this implies that if $F \in E(K)$ we need to have $F(t+a) = F(t) F(a)$. This is true to all a for which $1 > F(a) > 0$. Since F is continuous and a solution of that functional equation, but all these solutions are of the form $F(t) = \exp(-st)$ for some $s \in \mathbb{R}_+$. Then if $F \in K$ there exists a probability measure μ on $\mathbb{R}_+$ such that

$$F(t) = \int_{\mathbb{R}_+} e^{-st} \mu(\mathrm{d}s).$$

This proves the key claim in the theorem if F is bounded and $F \in K$. However is clear that if F is totally monotone, then $F \in \mathscr{C}$ and if $0 < F(0+) < \infty$ we have that $F(t) / F(0+)$ is bounded and >0 and in K. □

Lemma 12. *For any* $h \in \mathscr{H}$ *and* $t \geqslant 0$*,*

$$F_K(t, h) = \int_{\mathbb{R}_+} e^{-tx} \mu^h(\mathrm{d}x)$$

where $\mu^h(\mathbb{R}_+) = \|h\|^2$.

Proof. F_K is bounded because $|F_K(t, h)| \leqslant \|Kh\| \, \|h\| \leqslant \|h\|^2$ and totally monotone, so it has this representation note that $F(0, h) = \|h\|^2$. □

Lemma 13. *There is only one* C^* *representation* X_0 *of* $C^0_\infty(\mathbb{R}_+, \mathbb{C})$ *such that*

$$X_0(e^{-t \cdot}) = K(t)$$

Proof. Consider the set $\mathscr{E} = \mathrm{span}_{\mathbb{C}}\{e^{-tx}, t \geqslant 0\} \subset C^0_\infty$. Moreover $\mathscr{E}$ is a $*$-subalgebra on C^0_∞ and we define

$$X_{00}: \mathscr{E} \to \mathscr{B}(\mathscr{H})$$

as $X_{00}(e^{-tx}) = K(t)$ and then extend by linearity to all $\mathscr{E}$. X_{00} is a $*$-homomorphism since K is a semigroup. Moreover for $f = \sum_i \lambda_i e^{-t_i x}$ we have

$$\langle h, X_{00}(f) \, h \rangle = \sum_i \lambda_i F_K(t_i, h) = \sum_i \lambda_i \int_{\mathbb{R}_+} e^{-t_i x} \mu^h(\mathrm{d}x) = \int_{\mathbb{R}_+} f(x) \, \mu^h(\mathrm{d}x)$$

so by using that $X_{00}(f)$ is self-adjoint

$$|\langle X_{00}(f) h, X_{00}(f) \, h \rangle| = |\langle h, X_{00}(f^2) \, h \rangle| \leqslant \|f^2\|_\infty \|h\|^2 = \|f\|^2_\infty \|h\|^2,$$

and , we have that $\|X_{00}(f)\| \leqslant \|f\|_\infty$. As exercise we leave to prove that $\mathscr{E}$ is dense in $C^0_\infty(\mathbb{R}_+, \mathbb{C})$ (Stone-Weierstrass and a localization argument). Then we can extend X_{00} from $\mathscr{E}$ to C^0_∞ by continuity with the operator norm. □