Lecture 19 – 2020.6.23 – 14:15 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

We continue the proof of the last lecture.

Theorem 1. *Let $(K(t))_{t\geqslant 0}$ be a strongly continuous semigroup of self-adjoint contractions. There exists a unique C^*-homomorphism $X: C_b^0(\mathbb{R}_+;\mathscr{C})\to\mathscr{B}(\mathscr{H})$ such that*

1. $X(e^{-t\cdot})=K(t)$
2. *if $f_n\to f$ pointwise and $\sup_n\|f_n\|<\infty$, then $X(f_n)\to X(f)$ weakly.*

Last time we proved that:

1. There exists a unique $*$-homomorphism $X: C_\infty^0(\mathbb{R}_+,\mathbb{C})\to\mathscr{B}(\mathscr{H})$ where $C_\infty^0(\mathbb{R}_+,\mathbb{C})$ is the set of continuous functions going to zero at infinity.
2. For any $h\in\mathscr{H}$ there exists a unique positive measure μ^h on $\mathbb{R}_+$ such that $\mu^h(\mathbb{R}_+)=\|h\|^2$ and
$$\langle K(t)h,h\rangle=\int_{\mathbb{R}_+} e^{-tx}\mu^h(\mathrm{d}x).$$
3. For any $f\in C_\infty^0(\mathbb{R}_+,\mathbb{C})$ we have
$$\langle X(f)h,h\rangle=\int_{\mathbb{R}_+} f(x)\,\mu^h(\mathrm{d}x).$$

We introduce a measure
$$\mu^{h_1,h_2}:=\frac{1}{4}\sum_{k=0}^{3} i^k\mu^{h_1+(i)^k h_2}$$
by polarisation and we have
$$\langle X(f)h_1,h_2\rangle=\int_{\mathbb{R}_+} f(x)\,\mu^{h_1,h_2}(\mathrm{d}x).$$

Lemma 2. *We have that*
$$\frac{\mathrm{d}\mu^{X(f)h_1,h_2}}{\mathrm{d}\mu^{h_1,h_2}}=f(x)$$

Proof. The measure μ^{h_1,h_2} can be characterised by
$$\langle K(t)h_1,h_2\rangle=\int_{\mathbb{R}_+} e^{-tx}\mu^{h_1,h_2}(\mathrm{d}x)$$
and we have
$$\langle K(t)X(f)h_1,h_2\rangle=\int_{\mathbb{R}_+} e^{-tx}\mu^{X(f)h_1,h_2}(\mathrm{d}x)=\int_{\mathbb{R}_+} e^{-tx}f(x)\,\mu^{h_1,h_2}(\mathrm{d}x)$$
so by identification of Laplace transforms we have the claim. □

Proof. (of Theorem 1) Define the linear operator $\tilde{X}(f)$ by
$$\langle\tilde{X}(f)h_1,h_2\rangle=\int_{\mathbb{R}_+} f(x)\,\mu^{h_1,h_2}\mathrm{d}x$$

for all $h_1, h_2 \in \mathcal{H}$. We have

$$\|\tilde{X}(f)\|_{\mathcal{B}(\mathcal{H})} = \sup_{\|h_1\|=\|h_2\|=1} \left|\int_{\mathbb{R}_+} f(x)\, \mu^{h_1,h_2}(\mathrm{d}x)\right| \leqslant \|f\|_\infty \sup_{\|h_1\|=\|h_2\|=1} |\mu^{h_1,h_2}(\mathbb{R}_+)| \lesssim \|f\|_\infty$$

so $\tilde{X}(f)$ is bounded. Moreover one can show easily that $\langle \tilde{X}(f)h_1, h_2\rangle = \langle h_1, \tilde{X}(f^*)h_2\rangle$. The approximation property is quite easy to prove since if $f_n \to f$ pointwise and the family is bounded then by dominated convergence

$$\langle \tilde{X}(f_n)h_1, h_2\rangle = \int_{\mathbb{R}_+} f_n(x)\, \mu^{h_1,h_2}\mathrm{d}x \to \int_{\mathbb{R}_+} f(x)\, \mu^{h_1,h_2}\mathrm{d}x = \langle \tilde{X}(f)h_1, h_2\rangle$$

so we have weak convergence. Moreover if $f \in C_b^0(\mathbb{R}_+)$ then there exists $(f_n)_{n\geqslant 0} \subset C_\infty^0(\mathbb{R}_+)$ such that $f_n \to f$ pointwise and $\sup_n \|f_n\| < \infty$ (simply by multiplying f with a sequence of dilations of a given bounded functions of compact support). So there can be only one such operator which extends X from C_∞^0. We have to prove that $\tilde{X}$ is an homomorphism. Take $f, g \in C_b^0(\mathbb{R}_+, \mathbb{C})$ and consider two approximating sequences $(f_n)_n, (g_n)_n \subseteq C_\infty^0(\mathbb{R}_+)$ then taking $n \to \infty$

$$\langle \tilde{X}(fg_m)h_1, h_2\rangle \leftarrow \langle \tilde{X}(f_n g_m)h_1, h_2\rangle = \langle \tilde{X}(f_n)\tilde{X}(g_m)h_1, h_2\rangle \to \langle \tilde{X}(f)\tilde{X}(g_m)h_1, h_2\rangle$$

so taking $m \to \infty$ we get $\langle \tilde{X}(fg)h_1, h_2\rangle = \langle \tilde{X}(f)\tilde{X}(g)h_1, h_2\rangle$. This concludes the proof by taking $X = \tilde{X}$. □

Now we have seen that if $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group this is equivalent to have an representation X_U of $C_b^0(\mathbb{R}, \mathbb{C})$ in $\mathcal{B}(\mathcal{H})$ and if $(K(t))_{t\geqslant 0}$ is a self-adjoint, strongly continuous contraction semigroup, then we have a representation X_K of $C_b^0(\mathbb{R}_+, \mathbb{C})$ on $\mathcal{B}(\mathcal{H})$. We want to look into the relation between these two objects.

Definition 3. *We say that $(U(t))_{t\in\mathbb{R}}$ (as before) has positive energy for each $f \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $\mathrm{supp}(f) \subseteq (-\infty, 0)$ we have that $X_U(f) = 0$.*

Remark 4. Assume that $f_1, f_2 \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $f_1 = f_2$ on $[0, \infty)$ then if U has positive energy then $X_U(f_1) = X_U(f_2)$.

Lemma 5. *U has positive energy iff for any $h \in \mathcal{H}$ μ_U^h is supported on $\mathbb{R}_+ = [0, \infty)$.*

Proof. $\langle X_U(f)h_1, h_2\rangle = \int_\mathbb{R} f(x)\, \mu^h(\mathrm{d}x)$ if the measure is supported on $\mathbb{R}_+$ then $X(f) = 0$ if $\mathrm{supp}(f) \subseteq \mathbb{R}_{<0}$. On the other hand if $\mathrm{supp}(f) = (-\infty, 0)$ then $\int_\mathbb{R} f(x)\, \mu^h(\mathrm{d}x) = 0$ from which we get that $\mathrm{supp}(\mu^h) \subseteq \mathbb{R}_+$. □

Remark 6. If $(U(t))_{t\in\mathbb{R}}$ has positive energy and $g \in C_b^0(\mathbb{R}_+, \mathbb{C})$ then we can define $X_U(g)$ in a unique way as follows: we take $\tilde{g} \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $\tilde{g} = g$ on $\mathbb{R}_+$ and we define $X_U(g) = X_U(\tilde{g})$. This definition is a good one since the value do not depends on the extension $\tilde{g}$, indeed if $\hat{g}$ is another extension then $\tilde{g} - \hat{g}$ is supported on $(-\infty, 0)$ and $X_U(\hat{g}) = X_U(\tilde{g})$.

Theorem 7. *Assume $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group with positive energy, then $K(t) = X_U(e^{-t\cdot})$ is a strongly continuous self-adjoint contraction semigroup and also $X_U = X_K$ on $C_b^0(\mathbb{R}_+, \mathbb{C})$. The converse is true, i.e. if we have K and we define $U(t) = X_K(e^{it\cdot})$, then $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group with positive energy and $X_K = X_U$.*

Proof. From $e^{-t_1 s}e^{-t_2 s} = e^{-(t_1+t_2)s}$ we have $K(t_1)K(t_2) = K(t_1 + t_2)$ and the other properties follows easily, moreover by dominated convergence $\langle h_1, K(t)h_2\rangle \to \langle h_1, K(s)h_2\rangle$ if $t \to s$ and strong continuity follows since K is a contraction, i.e. $\|K(t)h\|^2 = \langle h, K(2t)h\rangle \leqslant \|e^{-2t\cdot}\|_{C_b^0(\mathbb{R}_+)}\mu^h(\mathbb{R}_+) = \|h\|^2$. The reverse implication is left as exercise. □

We want to justify now the name of "positive energy". This is not fundamental in the following but will give a better grasp of the connection with standard physical intuition.

Let $\mathscr{D}_H$ be a subspace of $\mathscr{H}$ such that $h \in \mathscr{D}_H$ iff $t \mapsto U(t)h$ is strongly differentiable in 0. For any $h \in \mathscr{D}_H$ we define

$$Hh = \frac{1}{i}\lim_{t\to 0}\frac{U(t)h-h}{t} \in \mathscr{H}.$$

Is simple to prove that H is a linear operator $H\colon \mathscr{D}_H \to \mathscr{H}$. For generic U, the operator H is not bounded, which implies that H cannot be extended as a continuous operator on all $\mathscr{H}$. H is an *unbounded operator* and $\mathscr{D}_H$ is called the domain of H.

Lemma 8. *$h \in \mathscr{D}_H$ iff*

$$\int_{\mathbb{R}} x^2 \mu^{h,U}(\mathrm{d}x) < \infty, \quad \textit{and then} \qquad \|Hh\|^2 = \int_{\mathbb{R}} x^2 \mu^{h_1,h_2,U}(\mathrm{d}x).$$

If $h_1 \in \mathscr{D}_H$ and $h_2 \in \mathscr{H}$ then

$$\int_{\mathbb{R}} |x||\mu^{h_1,h_2,U}|(\mathrm{d}x) < \infty, \quad \textit{and} \qquad \langle Hh_1, h_2\rangle = \int_{\mathbb{R}} x\mu^{h_1,h_2,U}(\mathrm{d}x).$$

Proof. Step 1. For any $h_1 \in \mathscr{D}_H$ and $h_2 \in H$

$$\int_{\mathbb{R}} |x||\mu^{h_1,h_2,U}|(\mathrm{d}x) = \sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1} \int_{\mathbb{R}} xf(x)\,\mu^{h_1,h_2,U}(\mathrm{d}x) = \sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1} \langle X(xf(x))h_1, h_2\rangle$$

$$\leqslant \|h_2\|_H \Big(\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1} \|X(xf(x))h_1\|\Big)^{1/2} \leqslant \|h_2\|_H \left(\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1} \int_{\mathbb{R}} (xf(x))^2 \mu^{h_1,h_1,U}(\mathrm{d}x)\right)^{1/2} \leqslant C_{h_1}\|h_2\|_H$$

But this implies that there exists h_1' such that $\langle h_1', h_2\rangle = \int_{\mathbb{R}} x\mu^{h_1,h_2,U}(\mathrm{d}x)$. Now we want to prove that $h_1' = Hh_1$

$$\left\langle \frac{1}{it}(U(t)-1)h - h_1', \frac{1}{it}(U(t)-1)h - h_1'\right\rangle = \left\|\frac{1}{it}(U(t)-1)h\right\|^2 + \|h_1'\|^2 - 2\mathrm{Re}\left\langle \frac{1}{it}(U(t)-1)h, h_1'\right\rangle$$

$$= \int_{\mathbb{R}} \underbrace{\left(2\frac{1-\cos(tx)}{t^2} + x^2 - 2\frac{\sin(tx)}{t}x\right)}_{G(t,x)} \mu^{h_1}(\mathrm{d}x)$$

Now $|G(t,x)| \leqslant Cx^2$ is uniformly bounded and pointwise converge to zero as $t \to 0$, so by Lebesgue dominated convergence we conclude that this quantity goes not zero. So we have that if $\int x^2\mu^h(\mathrm{d}x) < \infty$ we have that $U(t)h$ is strongly differentiable in zero. On the other hand, if $U(t)h$ is strongly differentiable then

$$\sup_{t\in(-1,1)} \left\|\frac{1}{it}(U(t)-1)h\right\|^2 = C < \infty$$

and in particular

$$\int x^2\mu^h(\mathrm{d}x) = 2\int \liminf_{t\to 0} \frac{1-\cos(tx)}{t^2}\mu^h(\mathrm{d}x) \leqslant \liminf_{t\to 0} 2\int \frac{1-\cos(tx)}{t^2}\mu^h(\mathrm{d}x) = \liminf_{t\to 0}\left\|\frac{1}{it}(U(t)-1)h\right\|^2 < C.$$

The rest of the proof is left as exercise. □

Theorem 9. *$\mathscr{D}_H$ is dense in $\mathscr{H}$ and $h_1, h_2 \in \mathscr{D}(H)$ we have $\langle Hh_1, h_2\rangle = \langle h_1, Hh_2\rangle$, so H is symmetric*

Proof. If $h \in \mathscr{H}$ define $h_\ell = \int_0^\ell U(s) h \mathrm{d}s$ we prove that $h_\ell \in \mathscr{D}_H$: indeed

$$\frac{\mathrm{d}\mu^{h_\ell}}{\mathrm{d}\mu^h} = \frac{1}{x^2}(e^{ihx}-1)(e^{-ihx}-1)$$

then

$$\int x^2 \mu^{h_\ell}(\mathrm{d}x) \leqslant C \int \mu^h(\mathrm{d}x) < \infty$$

and $h_\ell \in \mathscr{D}_H$.

$$\int e^{itx} \mu^{h_\ell}(\mathrm{d}x) = \left\langle U(t) \int_0^\ell U(s_1) h \mathrm{d}s_1, \int_0^\ell U(s_2) h \mathrm{d}s_2 \right\rangle = \int_{[0,\ell]^2} \int_{\mathbb{R}} e^{i(t+s_1+s_2)x} \mu^h(\mathrm{d}x) \mathrm{d}s_1 \mathrm{d}_2$$

and by Fubini we can exchange the integrals and obtain

$$\int e^{itx} \mu^{h_\ell}(\mathrm{d}x) = \int e^{itx} \frac{1}{x^2}(e^{ihx}-1)(e^{-ihx}-1) \mu^h(\mathrm{d}x)$$

and by identification of Fourier transforms. We have $\|h_\ell/\ell - h\| \to 0$ as $\ell \to 0$, we have

$$\|h_\ell/\ell - h\|^2 = \left\| \frac{1}{\ell} \int_0^\ell (U(s)-1) h \mathrm{d}s \right\|^2 \leqslant \sup_{s \in [0,\ell]} \|(U(s)-1)h\| = o(\ell)$$

by strong continuity. The symmetry is quite simple since

$$\langle H h_1, h_2 \rangle = \lim_{t \to 0} \left\langle \frac{U(t)-1}{it} h_1, h_2 \right\rangle = \lim_{t \to 0} \left\langle h_1, \frac{U(-t)-1}{-it} h_2 \right\rangle = \langle h_1, H h_2 \rangle.$$

□

Remark 10. Is possible to prove that $(H, \mathscr{D}_H)$ is self-adjoint, i.e. $H^* = H$. (given the natural definition of the adjoint of a densely defined unbounded operator)

If $h_1, h_2 \in \mathscr{D}_H$ we define $\mathscr{E}(h_1, h_2) = \langle H h_1, h_2 \rangle$. If $h_1 \in \mathscr{D}_H$ and $\|h\|_{\mathscr{H}} = 1$ then we define $\mathscr{E}(h, h)$ to be the energy of the state $h \in \mathscr{H}$.

Recall that $(\mathscr{H}, \mathscr{A}, Q_0)$ is our quantum space and if $h \in \mathscr{H}$ gives the vector state $\omega^h(a) = \langle Q_0(a) h, h \rangle$. So the energy is an extension of this formula for the unbounded operator H which formally is the derivative of the time-evolution group U. We had $Q_t(a) = U(-t) Q_0(a) U(t)$. If it is possible to take the derivative wrt. to t then we obtain

$$\partial_t Q_t(a) = \frac{1}{i}[H, Q_t(a)]$$

(this has to justified).

We have that $(h_1, h_2) \mapsto \mathscr{E}(h_1, h_2)$ is an Hermitian form (i.e. linear in the first and antilinear in the second variable).

Theorem 11. *The form $\mathscr{E}(h_1, h_2)$ is non-negative definite iff $(U(t))_{t \in \mathbb{R}}$ has positive energy.*

(to be continued)