Lecture 2 – Wed April 22nd 2020 – 12:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

## Mathematical model of Classical and Quantum Mechanics

- *Observables*. $A \in \mathscr{O}$ (e.g. components of magnetic moment, position, speed/momentum, energy) Connected with some measuring apparatus which has a scale where you read a real number. More observables can be constructed from $A$: $\lambda A, A^n \in \mathscr{O}$ $\lambda \in \mathbb{R}$. $A^n A^m = A^{n+m}$. $A \geqslant 0 \Leftrightarrow A = B^2, B \in \mathscr{O}$.

- *States*. $\omega \in \mathscr{S}$. (the state of the system: e.g. the state of the atoms in the Stern–Gerlach experiment beam, the state of a particle in motion in a particle accellerator, "the state of world") You know that different states exists because when we measure an observable we get different numbers. This relation between measurments on states and values of observables is "statistical" in the sense that $\omega(A) = \langle \omega, A \rangle \in \mathbb{R}$ represent the measuring of $A$ on the state $\omega$, has to be considered as an average over "experiences".

$$\omega(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} m_{\omega}^{(i)}(A).$$

$\omega : \mathscr{O} \to \mathbb{R}$. $\omega \equiv \{\omega(A) : A \in \mathscr{O}\}$.

We have the following relations between states and observables:

$$\omega(A) = \omega'(A), \forall A \in \mathscr{O} \Leftrightarrow \omega = \omega'$$

$$\omega(A) = \omega(B), \forall \omega \in \mathscr{S} \Leftrightarrow A \approx B$$

$$\omega(\lambda A) = \lambda \omega(A), \qquad \omega(A^n + A^m) = \omega(A^n) + \omega(A^m)$$

$$\omega(A^0) = 1 \Rightarrow A^0 = 1, \omega(1) = 1. \qquad \text{states are } \textit{normalized}.$$

$$A \geqslant 0 \Rightarrow A = B^2 \Rightarrow \omega(A) = \omega(B^2) \geqslant 0 \qquad \text{states are } \textit{positive}.$$

Norm on $\mathscr{O}$:

$$\|A\| = \sup_{\omega \in \mathscr{S}} |\omega(A)|$$

$$\|\lambda A\| = |\lambda| \|A\|, \quad \|A\| = 0 \Rightarrow A = 0.$$

We have also $\|A^2\| = \|A\|^2$. Indeed

$$\omega(\|A\| \pm A) = \|A\| \pm \omega(A) \geqslant 0 \text{“}\Rightarrow\text{”} \|A\| \pm A \geqslant 0.$$

$$\|A\|^2 - A^2 = (\|A\| + A)(\|A\| - A) \geqslant 0 \Rightarrow \omega(\|A\|^2 - A^2) \geqslant 0 \Rightarrow \|A\|^2 - \omega(A^2) \geqslant 0.$$

On the other hand

$$0 \leqslant (\|A\| \pm A)^2 = \|A\|^2 + A^2 \pm 2\|A\|A \Rightarrow 2\|A\||\omega(A)| \leqslant \|A\|^2 + \omega(A^2) \leqslant \|A\|^2 + \|A^2\|$$

taking sup over $\omega$ in $2\|A\||\omega(A)| \leqslant \|A\|^2 + \|A^2\|$ we get $\|A\|^2 \leqslant \|A^2\|$.

The states induce a linear structure over $\mathcal{O}$: we can define a new observable $C$ by doing

$$\omega(C) = \omega(A) + \omega(B)$$

for given $A, B \in \mathcal{O}$. We can extend $\mathcal{O}$ to a linear space and

$$\|A + B\| \leqslant \|A\| + \|B\|.$$

So at this point if we assume completeness we will have a Banach algebra, but we are still not accounting for sequentials measurements. What about $AB$? See in Strocchi the discussion on this point at page 19, working with *Jordan algebras*

$$A \circ B = \frac{1}{2}[(A+B)^2 - A^2 - B^2]$$

(not associative in general).

**Crucial techinical assumption.** $\mathcal{O} \subseteq \mathcal{A}$ where $\mathcal{A}$ (non-commutative) algebra over $\mathbb{C}$ with involution $A \mapsto A^*$

$$(\lambda A + \beta B)^* = \bar{\lambda} A^* + \bar{\beta} B^*, \qquad (AB)^* = B^* A^*$$

$$\forall A \in \mathcal{A}, \quad A^* A \geqslant 0, \qquad \omega(A^* A) \geqslant 0 \qquad \omega \in \mathcal{S}$$

$$\|AB\| := \sup_{\omega \in \mathcal{S}} |\omega(AB)| \leqslant \|A\|\,\|B\|. \qquad \boldsymbol{\|A^* A\| = \|A\|\,\|A^*\|.}$$

One simple consequence: take $\lambda \in \mathbb{R}$

$$0 \leqslant \omega((\lambda A + 1)^* (\lambda A + 1)) = \lambda^2 \omega(A^* A) + \lambda\,\omega(A^*) + \lambda\,\omega(A) + 1$$

then $\omega(A^*) = \overline{\omega(A)}$ and from this we have $\|A^*\| = \|A\|$.

**Mathematical model for a physical system.**

A physical system is the given of observables and states,

- Observables form a $C^*$-*algebra* $\mathcal{A}$ *with unity*.
- States $\mathcal{S}$ are normalized positive linear functionals on $\mathcal{A}$. We assume the set of states to be *full* (i.e. it separates the observables). Moreover observables should separate states (but this is by definition). Usually $\mathcal{S}$ is only a subset of all the positive linear functionals.

**Example.** Classical mechanical system $(q, p) \in \Gamma \subseteq T^*\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ where $q$ is position and $p$ momentum. The set of observables are the (continuous) functions $\mathscr{A} = C(\Gamma, \mathbb{C})$ $f^*(q,p) = \overline{f(q,p)}$. The states are (a subset of) the probability measures on $\Gamma$:

$$\omega(f) = \int_\Gamma f(q,p)\,\omega(\mathrm{d}q \times \mathrm{d}p).$$

$$\|f\| = \sup_{\omega \in \mathscr{S}} |\omega(f)|.$$

In classical physics one assume that states of the form $\omega = \delta_{(q_0,p_0)}$ are possible, these states are characterised by the fact that the dispersion

$$\Delta_\omega(f) = [\omega(f^2 - \omega(f)^2)]^{1/2} \geqslant 0,$$

is zero for all observables.