Lecture 20 – 2020.6.24 – 8:30 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Let $(U(t))_{t\in\mathbb{R}}$ be a strongly continuous unitary group, define $\mathscr{D}_H\subset\mathscr{H}$ as $h\in\mathscr{D}_H$ if $U(t)h$ is strongly differentiable for $t=0$. We defined the Hamiltonian

$$H(h)=\frac{1}{i}\lim_{t\to 0}\frac{U(t)h-h}{t}$$

We proved that $\mathscr{D}_H$ is dense in H and $h_1\in\mathscr{D}_H$ iff $\int x^2\mu^{h,U}(\mathrm{d}x)<\infty$.

The energy $\mathscr{E}$ is the form $\mathscr{E}(h_1,h_2)=\langle Hh_1,h_2\rangle$ for $h_1,h_2\in\mathscr{D}_H$ (but one can allow $h_2\in\mathscr{H}$). We proved that $\mathscr{E}(h_1,h_2)=\overline{\mathscr{E}(h_1,h_2)}$, and the form is Hermitian.

Theorem 1. *U has positive energy iff $\mathscr{E}(h,h)\geqslant 0$ for all $h\in\mathscr{D}_H$.*

Proof. If U has positive energy, we saw in the last lecture that μ^h is supported in $\mathbb{R}_+$ and we have

$$\mathscr{E}(h,h)=\int_{\mathbb{R}}x\,\mu^{h,U}(\mathrm{d}x)=\int_{\mathbb{R}_+}x\,\mu^{h,U}(\mathrm{d}x)\geqslant 0.$$

Assume now that $\mathscr{E}$ is non-negative definite and assume that U has not positive energy, therefore there exists $h\in\mathscr{H}$ such that μ^h has some support on $(-\infty,0)$. We can assume that $\operatorname{supp}(\mu^h)\subset(-\infty,-c)$ for some $c>0$ since we can consider the vector $X_U(f)h$ with $\operatorname{supp}(f)\subset(-\infty,-c)$ and $\mathrm{d}\mu^{X(f)h}=f\mathrm{d}\mu^h$. So now taking $h_\ell=\int_0^\ell U(s)h\mathrm{d}s$ and

$$\mu^{h_\ell}(\mathrm{d}x)=\frac{1}{x^2}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x).$$

Let $d>c$ such that $\mu([-d,-c])>0$. Note that $h_\ell\in\mathscr{D}_H$ and

$$\mathscr{E}(h_\ell,h_\ell)=\int_{\mathbb{R}}x\,\mu^{h_\ell}(\mathrm{d}x)=\int_{\mathbb{R}}x\frac{1}{x^2}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x)<\int_{[-d,-c]}\frac{1}{x}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x)$$

and if ℓ is small enough this quantity is negative. □

Recall the definitions

$$F_U(t,h)=\langle U(t)h,h\rangle=\int_{\mathbb{R}}e^{itx}\mu^{h,U}(\mathrm{d}x),$$

$$F_K(t,h)=\langle K(t)h,h\rangle=\int_{\mathbb{R}_+}e^{-tx}\mu^{h,K}(\mathrm{d}x).$$

Theorem 2. *The function F_K is holomorphic when $t\in\mathbb{C}$ and $\operatorname{Re}(t)>0$ and it is continuous when $\operatorname{Re}(t)\geqslant 0$. Moreover, we have that*

$$F_U(s,h)=F_K(is,h)=\lim_{y\downarrow 0}F_K(is+y,h).$$

Proof. If $\operatorname{Re}(t_1)>0$ take $\varepsilon\in\mathbb{C}$ with $|\varepsilon|<\operatorname{Re}(t_1)$ then

$$|F(t_1+\varepsilon,h)|=\left|\int_{\mathbb{R}_+}e^{-t_1x}e^{-\varepsilon s}\mu^{h,K}(\mathrm{d}x)\right|\leqslant\int_{\mathbb{R}_+}e^{-\operatorname{Re}(t_1)x}e^{-|\varepsilon|s}\mu^{h,K}(\mathrm{d}x)<\infty,$$

and by monotone convergence the series

$$\sum_n |\varepsilon|^n \int_{\mathbb{R}_+} e^{-t_1 x} \frac{x^n}{n!} \mu^{h,K}(\mathrm{d}x)$$

is convergent and so F has a convergent power series expansion in the claimed domain and continuity derives from the dominated convergence theorem. Moreover

$$\lim_{y\downarrow 0} F_K(is+y,h) = \int_{\mathbb{R}_+} e^{isx} \mu^{h,K}(\mathrm{d}x) = F_U(s,h)$$

when U is defined so that $\mu^{h,K} = \mu^{h,U}$. □

Remark 3. We can define the generator H' of K similarly as we defined the generator H of U. Namely $\mathscr{D}_{H'}$ is defined as the set of vectors $h \in \mathscr{H}$ such that $K(t)h$ is strongly differentiable in zero and define

$$H'h = -\lim_{t\downarrow 0} \frac{K(t)h - h}{t}.$$

But if U and K are related so that $X_U = X_K$ then $H' = H$ and $\mathscr{D}_H = \mathscr{D}_{H'}$.

Consider now $\mathscr{H} = L^2(\mathbb{R}^n, \mathrm{d}x)$. $\mathscr{A} = C_b^0(\mathbb{R}^n, \mathbb{C})$ and $(Q_0(a)h)(x) = a(x)h(x)$. Define

$$K(t)h = \rho_t * h = \frac{1}{(2\pi t)^{n/2}} \int e^{-|x-y|^2/(2t)} h(y)\mathrm{d}y.$$

Theorem 4. $(K(t))_{t\geqslant 0}$ *is a strongy continuous, self-adjoint contraction semigroup.*

Proof. Let $\mathscr{F}(h) = \int_{\mathbb{R}^n} e^{ikx} h(x)\mathrm{d}x$ the Fourier transform. Recall Plancherel's theorem

$$\int_{\mathbb{R}^n} h_1(x)\overline{h_2(x)}\mathrm{d}x = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathscr{F}(h_1)(k)\overline{\mathscr{F}(h_2)}(k)\mathrm{d}k$$

and that $\mathscr{F}(a*b) = (\mathscr{F}a)(\mathscr{F}b)$. Moreover $\mathscr{F}(\rho_t)(k) = \exp(-t|k|^2/2)$. Now

$$\|K(t)h\|_{L^2}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathscr{F}(\rho_t * h)(k)|^2 \mathrm{d}k = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-t|k|^2)|\mathscr{F}(h)(k)|^2 \mathrm{d}k$$
$$\leqslant \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathscr{F}(h)(k)|^2 \mathrm{d}k = \|h\|_{L^2}^2$$

so K is a contraction. Moreover

$$\|K(t)h - h\|_{L^2}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 - \exp(-t|k|^2/2))^2 |\mathscr{F}(h)(k)|^2 \mathrm{d}k \to 0$$

as $t \to 0$, so it is strongly continuous. Additionally it is self-adjoint since

$$\langle K(t)h_1, h_2\rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-t|k|^2/2)^2 \mathscr{F}(h_1)(k)\overline{\mathscr{F}(h_2)(k)}\mathrm{d}k = \langle h_1, K(t)h_2\rangle$$

and the semigroup property derives from

$$\mathscr{F}(K(t)K(s)h)(k) = \exp(-t|k|^2/2)\exp(-s|k|^2/2)\mathscr{F}(h)(k) = \exp(-(t+s)|k|^2/2)\mathscr{F}(h)(k) = \mathscr{F}(K(t+s)h)(k).$$

□

Take $f \in C^\infty \cap L^p$ for any $p \geqslant 1$. Then in $L^2(\mathbb{R}^n)$ we have

$$\lim_{t\downarrow 0} \mathscr{F}\left(\frac{K(t)f - f}{t}\right)(k) = \lim_{t\downarrow 0} \frac{e^{-tk^2/2}-1}{t}\mathscr{F}(f)(k) = -k^2\mathscr{F}(f)(k) = \mathscr{F}(\Delta f)(k)$$

so $H = -\Delta$ and one can prove that $\mathscr{D}_H = H^2$. Moreover $\mathscr{E}(h,h) = \int_{\mathbb{R}^n} |\nabla h|^2 \mathrm{d}x \geqslant 0$. So the semigroup has positive energy (it was already clear from the fact that it is a contraction).

So now

$$F_K(t,h) = \int_{\mathbb{R}^{2n}} \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{n/2}} h(x)\overline{h(y)}\mathrm{d}x\mathrm{d}y$$

and for $h \in L^2 \cap L^1$ we have the explicit representation

$$F_U(s,h) = F_K(is,h) = \int_{\mathbb{R}^{2n}} \frac{e^{-|x-y|^2/2(is)}}{(2\pi is)^{n/2}} h(x)\overline{h(y)}\mathrm{d}x\mathrm{d}y$$

where $(i)^{n/2} = e^{\pi i n/4}$ given the kind of limit we had to perform. We conclude therefore that for $h \in L^2 \cap L^1$

$$(U(s)h)(x) = \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/2(is)}}{(2\pi is)^{n/2}} h(y)\mathrm{d}y.$$

This is the model of the free particle in $\mathbb{R}^n$, i.e. a particle not interacting with any external system. In this case $(U(t))_{t\in\mathbb{R}}$ is a unitary group on $L^2(\mathbb{R}^n)$ and the expectation of any observable $Q_t(a)$ on the state ω^h evolves according to the equation

$$\omega_t^h(a) = \langle Q_t(a)h, h\rangle = \langle U(-t)Q_0(a)U(t)h, h\rangle = \langle Q_0(a)U(t)h, U(t)h\rangle.$$

Definition 5. *Assume that U has positive energy, we say that $h_0 \in \mathscr{H}$ is a ground state for U iff $U(t)h_0 = h_0$.*

Theorem 6. *h_0 is a ground state for U iff one of the following equivalent conditions hold:*

1. $\mu^{h_0}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$
2. $K(t)h_0 = h_0$
3. *$h_0 \in \mathscr{D}_H$ and $Hh_0 = 0$*
4. *$h_0 \in \mathscr{D}_H$ and $\mathscr{E}(h_0,h_0) = 0$*

Proof. Exercise. □

Remark 7. The name ground state comes from the fact that h_0 is the state of minimal energy of the system (i.e. the zero energy, in our normalization).

Definition 8. *h_0 a cyclic ground state if $\mathrm{span}\{U(t_1)Q_0(a_1)U(t_2)Q_0(a_2)\cdots h_0\}$ is dense in $\mathscr{H}$.*

A cyclic ground state allows to reconstruct all the Hilbert space from expectations of time evolutions of observables.

Indeed any $\omega^h(Q_t(a))$ can then be approximated by linear combinations of expressions of the form

$$\langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

for suitable $t_1,\dots,t_n$ since we used the fact that h_0 is invariant under U.

Assume that we are given a cyclic ground state.

Wightman functions are defined as

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1,\dots,t_k) = \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

where $\mathbb{A}_k = (a_1,\dots,a_k) \in \mathscr{A}^k$.

Lemma 9. $\mathbb{W}_{k,\mathbb{A}_k}$ *is invariant wrt. to time translations, namely*

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1,\dots,t_k) = \mathbb{W}_{k,\mathbb{A}_k}(t_1+s,\dots,t_k+s)$$

for all $s \in \mathbb{R}$.

Proof. By invariance of the ground state we have

$$\begin{aligned}\mathbb{W}_{k,\mathbb{A}_k}(t_1,\dots,t_k) &= \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle\\ &= \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)U(s)h_0, U(s)h_0\rangle\\ &= \langle U(-s)Q_{t_1}(a_1)U(s)U(-s)\cdots U(-s)Q_{t_n}(a_n)U(s)h_0, h_0\rangle\end{aligned}$$

and since $U(-s)Q_{t_1}(a)U(s) = Q_{t_1+s}(a)$ we have the result. □

We observe also that we can define the (reduced) function

$$W_{k,\mathbb{A}_k}(\xi_1,\dots,\xi_{k-1}) = \mathbb{W}_{k,\mathbb{A}_k}(t, t+\xi_1,\dots,t_k+\xi_{k-1}) = \langle Q_0(a_1)U(\xi_1)Q_0(a_2)U(\xi_2)\cdots Q_0(a_k)h_0, h_0\rangle$$

for $\xi_1,\dots,\xi_{k_1} \in \mathbb{R}$.