Lecture 21 – 2020.6.29 – 14:15 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Continuing to study Wightman and Schwinger functions.

Recall the setting $(\mathscr{H}, \mathscr{A}, Q_0, U(t))$. Where $(U(t))_t$ is a positive energy strongly continuous unitary group, or equivalently $(K(t))_t$ a self-adjoint, strongly continuous, contraction semigroup. We saw that the given of U is equivalent to the given of K. Recall also that a *ground state* $h_0 \in \mathscr{H}$ is a cyclic vector, invariant under U, cyclic here means that the span of the vectors of the form

$$U(t_1)Q_0(a_1)U(t_2)\cdots Q_0(t_n)h_0$$

(or equivalently of the vectors $K(t_1)Q_0(a_1)K(t_2)\cdots Q_0(t_n)h_0$).

We introduced functions (Wightman functions)

$$\mathscr{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k) = \langle Q_{t_1}(a_1)Q_{t_2}(a_2)\cdots Q_{t_k}(a_k)h_0, h_0\rangle$$

where $Q_t(a) = U(t)Q_0(a)U(-t)$, and where $\mathbb{A}_k = (a_1,\ldots,a_k) \in \mathscr{A}^k$.

We proved already that

Theorem 1. *For any $k \in \mathbb{N}$, $\mathbb{A}_k \in \mathscr{A}^k$, $t_1,\ldots,t_k, t \in \mathbb{R}$ we have*

$$\mathscr{W}_{k,\mathbb{A}_k}(t_1+t,\ldots,t_k+t) = \mathscr{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k).$$

For this reason we introduced the reduced Wightman functions

$$W_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_{k-1}) = \langle Q_{t0}(a_1)U(\xi_1)Q_0(a_2)U(\xi_2)\cdots U(\xi_{k-1})Q_0(a_k)h_0, h_0\rangle$$

and we have the property that

$$\mathscr{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k) = W_{k,\mathbb{A}_k}(t_2-t_1,\ldots,t_k-t_{k-1}).$$

Definition 2. *We consider a set of functions $\tilde{W}_{k,\cdot}(\cdot)\colon \mathscr{A}^k \times \mathbb{R}^{k-1} \to \mathbb{C}$. We say that $\tilde{W}_{k,\mathbb{A}_k}$ satisfy Axiom W1 (compatibility conditions) if the following properties hold*

1. *$\mathbb{A}_k = (a_1,\ldots,a_k) \in \mathscr{A}^k$ and $(t_1,\ldots,t_{k-1}) \in \mathbb{R}^{k-1}$, we have*

   $$\tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_{k-1}) = \tilde{W}_{k-1,\tilde{\mathbb{A}}_{k-1}}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{k-1})$$

   where $\tilde{\mathbb{A}}_{k-1} = (a_1,\ldots a_{i-1}, a_i a_{i+1}, a_{i+2},\ldots,a_k) \in \mathscr{A}^{k-1}$.

2. *$\mathbb{A}_{k-1} = (a_1,\ldots,a_{k-1}) \in \mathscr{A}^{k-1}$ and $T_{k-1} = (t_1,\ldots,t_{k-1}) \in \mathbb{R}^{k-1}$, we have*

   $$\tilde{W}_{k,(a_1,\ldots,a_{i-1},1_{\mathscr{A}},a_i,\ldots,a_{k-1})}(t_1,\ldots,t_k) = \tilde{W}_{k-1,\mathbb{A}_{k-1}}(t_1,\ldots,t_{i-2},t_{i-1}+t_i,t_{i+1},\ldots,t_{k-1}).$$

3. *$\mathbb{A}_k = (a_1,\ldots,a_k) \in \mathscr{A}^k$ and $T_{k-1} = (t_1,\ldots,t_{k-1}) \in \mathbb{R}^{k-1}$ we have*

   $$\overline{\tilde{W}_{k,\mathbb{A}_k}(T_{k-1})} = \tilde{W}_{k,\theta\mathbb{A}_k}(\bar{\theta}(T_{k-1}))$$

where $\theta(\mathbb{A}_k) = (a_k^*, a_{k-1}^*, \dots, a_1^*)$ *and* $\bar{\theta}(T_{k-1}) = (-t_{k-1}, -t_{k-2}, \dots, -t_2, -t_1)$.

Lemma 3. *Reduced Wightman functions satisfy these compatibility conditions (i.e. Axiom W1).*

Proof. Easy exercise. □

Let now introduce the Fréchet space $\mathscr{S}(\mathbb{R}^k)$ (locally convex topological vector space) such that $f \in \mathscr{S}(\mathbb{R}^k)$ iff $f \in C^\infty(\mathbb{R}^k)$ and $\|f\|_{n,\alpha} = \sup_{x\in\mathbb{R}^k} |(1+|x|)^n \mathrm{D}^\alpha f(x)| < \infty$ where $n \geqslant 0$ and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_0^k$ with $\mathrm{D}^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}$. We can consider the dual $\mathscr{S}'(\mathbb{R}^k) = (\mathscr{S}(\mathbb{R}^k))^*$, that is the space of linear functionals $T \in \mathscr{S}(\mathbb{R}^k) \to \mathbb{C}$ such that there exists n, α for which $|T(f)| \leqslant C_T \|f\|_{n,\alpha}$. Recall also that the Fourier transform $\mathscr{F}: L^1(\mathbb{R}^k) \to C^0(\mathbb{R}^k)$ is defined by

$$\mathscr{F} f(y) = \int_{\mathbb{R}^k} e^{ik\cdot y} f(x) \mathrm{d}x$$

and such that $\mathscr{F}: \mathscr{S}(\mathbb{R}^k) \to \mathscr{S}(\mathbb{R}^k)$ and the map is continuous wrt. to the topology $\mathscr{S}(\mathbb{R}^k)$ and invertible with

$$\mathscr{F}^{-1} f(y) = \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} e^{-ik\cdot y} f(x) \mathrm{d}x.$$

Then if $T \in \mathscr{S}'(\mathbb{R}^k)$ we can define $\mathscr{F}(T) = T \circ \mathscr{F}^{-1}$.

Definition 4. $\tilde{W}_{k,\cdot}(\cdot)$ *satisfy Axiom W2 (i.e. it is a Fourier transform of a distribution with support in* $\mathbb{R}_+^{k-1}$*) if* $\tilde{W}_{k,\mathbb{A}_k}(t_1, \dots, t_{k-1})$ *is continuous in* $t_1, \dots, t_{k-1}$ *and* $\tilde{W}_{k,\mathbb{A}_k} = \mathscr{F}(T_{k,\mathbb{A}_k})$ *for some* $T_{k,\mathbb{A}_k} \in \mathscr{S}'$ *such that*

$$|T_{k,\mathbb{A}_k}(f_1 \otimes \cdots \otimes f_{k-1})| \leqslant C_k \prod_{\ell=1}^{k-1} \|f_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathscr{A}}. \tag{1}$$

What means that T has support on $\mathbb{R}_+^k$? This means that if $f \in \mathscr{S}(\mathbb{R}^k)$ and $\mathrm{supp}(f) \subset \mathbb{R}^{k-1} \backslash (\mathbb{R}_+)^{k-1}$ then $T(f) = 0$.

Remark 5. The equation (1) is equivalent to

$$\left| \int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1, \dots, t_{k-1}) \overline{g_1(t_1)} \cdots \overline{g_{k-1}(t_{k-1})} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \right| \lesssim \tilde{C}_k \prod_{\ell=1}^{k-1} \|\mathscr{F}^{-1}(g_\ell)\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathscr{A}}.$$

for $g_1, \dots, g_{k-1} \in \mathscr{S}(\mathbb{R})$. Indeed recall that $\mathscr{F}(T_{k,\mathbb{A}_k}) = \tilde{W}_{k,\mathbb{A}_k}$ and

$$\mathscr{F}(T_{k,\mathbb{A}_k})(g) = \langle \tilde{W}_{k,\mathbb{A}_k}, g \rangle = \int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1, \dots, t_{k-1}) \overline{g(t_1, \dots, t_k)} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

but $\mathscr{F}(T_{k,\mathbb{A}_k})(g) = T_{k,\mathbb{A}_k}(\mathscr{F}^{-1}(g))$ and calling $\mathscr{F}^{-1}(g) = f$ and from this one can conclude.

Lemma 6. *The Wightman functions satisfy Axiom W2.*

Proof. Recall that from U we can construct homomorphisms $X_U: C_b^0(\mathbb{R}, \mathbb{C}) \to \mathscr{B}(\mathscr{H})$ such that $X_U(e^{it\cdot}) = U(t)$ and which is strongly continuous with respect to pointwise sequential convergence in bounded sets. So for any $g \in \mathscr{S}(\mathbb{R})$ we can define $U_g = \int_{\mathbb{R}} g(t) U(t) \mathrm{d}t = X_U(\mathscr{F} g)$. Indeed

$$\langle X_U(\mathscr{F} g) h_1, h_2 \rangle = \int \mathscr{F} g(x) \mu^{h_1,h_2}(\mathrm{d}x) = \int \int g(t) e^{itx} \mathrm{d}t \mu^{h_1,h_2}(\mathrm{d}x) = \int g(t) \int e^{itx} \mu^{h_1,h_2}(\mathrm{d}x) \mathrm{d}t$$

$$= \int g(t) \langle U(t) h_1, h_2 \rangle \mathrm{d}t.$$

Now

$$\int W_{k,\mathbb{A}_k}(t_1,\dots,t_k)g_1(t_1)\cdots g_k(t_k) = \langle Q_0(a_1)U(t_1)Q_0(a_2)U(t_2)\cdots Q_0(a_k)h_0,h_0\rangle g_1(t_1)\cdots g_{k-1}(t_{k-1})\mathrm{d}t_1\cdots\mathrm{d}t_k$$

$$=\langle Q_0(a_1)U(g_1)Q_0(a_2)U(t_2)\cdots U(g_{k-1})Q_0(a_k)h_0,h_0\rangle$$

$$=\langle Q_0(a_1)X_U(\mathscr{F}(g_1))Q_0(a_2)U(t_2)\cdots X_U(\mathscr{F}(g_{k-1}))Q_0(a_k)h_0,h_0\rangle$$

which can be bounded by

$$\|Q_0(a_1)\|\cdots\|Q_0(a_k)\|\,\|X_U(\mathscr{F}(g_1))\|\cdots\|X_U(\mathscr{F}(g_{k-1}))\|$$

which then gives readily the result using the fact that U has positive energy so

$$\|X_U(f)\|\leqslant\|f\|_{L^\infty(\mathbb{R}_+)}.$$

□

Let us consider now our last axiom. Recall that we defined $\theta(\mathbb{A}_k)=(a_k^*,a_{k-1}^*,\dots,a_1^*)$ and $\bar{\theta}(T_{k-1})=(-t_{k-1}, -t_{k-2},\dots,-t_2,-t_1)$.

For $\mathbb{A}_{k_1}=(a_1,\dots,a_{k_1})$ and $\mathbb{A}'_{k_1}=(a'_1,\dots,a'_{k_2})$ then we let

$$\mathbb{A}_{k_1}\mathbb{A}'_{k_2}=(a_1,a_2,\dots,a_{k_1}a'_1,\dots,a'_{k_2})\in\mathscr{A}^{k_1+k_2-1}$$

Definition 7. *The functions* $\tilde{W}_{k,\cdot}(\cdot)\colon\mathscr{A}^k\times\mathbb{R}^{k-1}\to\mathbb{C}$ *satisfy Axiom W3 (Hilbert-space positivity) if for any* $k\in\mathbb{N}_0$ *and any* $j_1,\dots,j_k\in\mathbb{N}$*, any* $T_{n-1,j}=(t_{1,(n-1,j)},\dots t_{n-1,(n-1,j)})$ *and* $\lambda_{n,j}\in\mathbb{C}$ *and* $\mathbb{A}_{n,j}=(a_{1,(n,j)},\dots a_{n,(n,j)})\in\mathscr{A}^n$ *where* $j\leqslant j_n$ *and* $n\leqslant k$ *we have*

$$\sum_{n_1+n_2=1}^{k}\sum_{h_1=1}^{j_{n_1}}\sum_{h_2=1}^{j_{n_2}}\lambda_{n_1,h_1}\overline{\lambda_{n_2,h_2}}\tilde{W}_{n_1+n_2-1,\theta(\mathbb{A}_{n_2,h_2})\mathbb{A}_{n_1,h_1}}(\bar{\theta}(T_{n_2-1,h_2}),T_{n_1-1,h_1})\geqslant 0.$$

Example: if $k=1$ we have only

$$\sum_{h_1=1}^{j_1}\sum_{h_2=1}^{j_2}\lambda_{h_1}\bar{\lambda}_{h_2}\tilde{W}_{1,a_{h_2}^*a_{h_1}}=\sum_{h_1=1}^{j_1}\sum_{h_2=1}^{j_2}\lambda_{h_1}\bar{\lambda}_{h_2}\langle Q_0(a_{h_2}^*)Q_0(a_{h_1})h_0,h_0\rangle$$

$$=\left\langle\sum_{h_1=1}^{j_1}\lambda_{h_1}Q_0(a_{h_1})h_0,\sum_{h_2=1}^{j_2}\lambda_{h_2}Q_0(a_{h_2})h_0\right\rangle\geqslant 0.$$

Another example gives

$$0\leqslant\lambda\bar{\lambda}\tilde{W}_{2,(a_2^*,a_1^*a_1,a_2)}(t_1,-t_1)=\langle Q_0(a_2^*)U(t_1)Q_0(a_1^*a_1)U(-t_1)Q_0(a_2)h_0,h_0\rangle=\|Q_0(a_1)U(-t_1)Q_0(a_2)h_0\|^2$$

Lemma 8. *Wightman functions satisfy Axiom W3.*

Proof. Let

$$H=\sum_{n_1=1}^{k}\sum_{h_1=1}^{j_{n_1}}\lambda_{n_1,h_1}Q_0(a_{1,(n_1,h_1)})U(t_{1,(n_1-1,h_1)})\cdots U(t_{n-1,(n_1-1,h_1)})Q_0(a_{n_1,(n_1,h_1)})h_0$$

and using $\langle H, H\rangle \geqslant 0$ and the definition of Wightman functions we get the claim. $\square$

Next lecture will be dedicated to giving the idea of the proof of equivalent properties $S1, S2, S3$ for the Schwinger functions (which are like Wightman functions but with K in place of U) and then we prove that if we are given functions $W1, W2, W3$ or $S1, S2, S3$ then we can come back and obtain the data of $(\mathscr{H}, Q_0, U)$ or $(\mathscr{H}, Q_0, K)$ of the Hilbert space, representations Q_0, K or U.