Lecture 22 – 2020.7.1 – 8:30 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

In the last lecture we introduced the reduced Wightman functions $(W_{k,\mathbb{A}_k})_k$ and show that they statisfy three basic properties

a) W1 – compatibility condition (encodes the fact that Q_0 is a C^*-representation and that U is a unitary group)

b) W2 – tempered distribution axiom (encodes the fact that U is strongly continuous with positive energy)

c) W3 – Hilbert space positivity (encodes the fact that the scalar product is Hermitian and positive)

Definition 1. *Schwinger functions, $k\in\mathbb{N}$ and $\mathbb{A}_k=(a_1,\ldots,a_k)\in\mathscr{A}^k$, $t_1,\ldots,t_{k-1}\geqslant 0$ and let*

$$S_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})=\langle Q_0(a_1)K(t_1)Q_0(a_2)K(t_2)\cdots K(t_{k-1})Q_0(a_k)h_0,h_0\rangle.$$

Recall that $\theta(\mathbb{A}_k)=(a_k^*,\ldots,a_1^*)$ and $\bar{\theta}(T_{k-1})=(-t_{k-1},-t_{k-2},\ldots,-t_1)$. We introduce now also another map on times as $\hat{\theta}(T_{k-1})=(t_{k-1},t_{k-2},\ldots,t_1)$. We will need also the composition $\mathbb{A}_k\cdot\mathbb{A}'_{k'}=(a_1,\ldots,a_ka'_1,\ldots a'_{k'})$.

Definition 2. *We say that the set of functions $(\tilde{S}_k\colon\mathscr{A}^k\times\mathbb{R}_+^{k-1}\to\mathbb{C})_k$ satisfy the axiom S1 (or compatibility condition)*

1. $\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_{k-1})=\tilde{S}_{k-1,\tilde{\mathbb{A}}_k}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{k-1})$ *where* $\tilde{\mathbb{A}}_k=(a_1,\ldots,a_ia_{i+1},\ldots,a_k)$ *and*

$$\tilde{S}_{k,(a_1,\ldots,\lambda a_i+\mu b_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\lambda\tilde{S}_{k,(a_1,\ldots,a_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})+\mu\tilde{S}_{k,(a_1,\ldots,b_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})$$

2. $\tilde{S}_{k,(a_1,\ldots a_{i-1},1,a_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\tilde{S}_{k-1,\mathbb{A}_{k-1}}(t_1,\ldots,t_{k-1})$

3. $\overline{\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})}=\tilde{S}_{k,\theta(\mathbb{A}_k)}(\hat{\theta}(T_k))$ *which is due to the fact that* $K(t)^*=K(t)$.

Lemma 3. *The Schwinger functions satisfy Axiom S1*

Let $T\in\mathscr{S}'(\mathbb{R}^{k-1})$ supported in $\mathbb{R}_+^{k-1}=(\mathbb{R}_+)^{k-1}$, i.e. $T(f)=T(f')$ when $f=f'$ on $\mathbb{R}_+^{k-1}$, i.e. functions which behave on $\mathbb{R}_+^{k-1}$ but arbitrarily elsewhere. For example $s\mapsto e^{-ts}$ belongs to $\mathscr{S}(\mathbb{R}_+)$ and

$$(s_1,\ldots,s_{k-1})\mapsto e^{-t_1s_1\cdots-t_{k-1}s_{k-1}}$$

is in $\mathscr{S}(\mathbb{R}_+^{k-1})$. We define the Laplace transform $\mathscr{L}(T)=G(t_1,\ldots,t_{k-1})$ as

$$G(t_1,\ldots,t_{k-1})=T((s_1,\ldots,s_{k-1})\mapsto e^{-t_1s_1\cdots-t_{k-1}s_{k-1}}).$$

If $f\in L^1$

$$\mathscr{L}(f)(t)=\int_0^\infty e^{-ts}f(s)\mathrm{d}s.$$

Definition 4. *Let $(\tilde{S}_k)_k$ as before. We say that they satisfy Axiom S2 (or that they are Laplace transform of a tempered distribution) if $\exists T_{k,\mathbb{A}_k}$ such that $\tilde{S}_{k,\mathbb{A}_k} = \mathscr{L}(T_{k,\mathbb{A}_k})$ and for all $g_1, \ldots, g_{k-1} \in \mathscr{S}(\mathbb{R}_+)$*

$$\left| \int_{\mathbb{R}_+^{k-1}} \tilde{S}_{k,\mathbb{A}_k}(t_1, \ldots, t_{k-1}) g(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \right| \leqslant \prod_{\ell=1}^{k-1} \|\mathscr{L} g_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k-1} \|a_k\|_{\mathscr{A}}. \tag{1}$$

Theorem 5. *The inequality (1) implies that $\tilde{S}_{k,\mathbb{A}_k}$ is the Laplace transform of a distribution.*

Proof. For a proof see the book of B. Simon "The $P(\varphi)_2$ Euclidean Quantum Field Theory", Chap. 2 Sect. 2.2. □

Lemma 6. *The Schwinger functions, satisfy Axiom S2.*

Proof. Similar to the analogous statement for Wightman functions. The essential step is to observe that

$$\int_{\mathbb{R}_+^{k-1}} S_{k,\mathbb{A}_k}(t_1, \ldots, t_{k-1}) g(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

$$= \langle Q_0(a_1) X_K(\mathscr{L} g_1) \cdots X_K(\mathscr{L} g_{k-1}) Q_0(a_k) h_0, h_0 \rangle$$

where X_K is the homomorphism generated by K as we introduced few lectures ago. □

Remark 7. We proved that $S_{k,\mathbb{A}_k} = \mathscr{L}(T_{k,\mathbb{A}_k})$, moreover $T_{k,\mathbb{A}_k}$ for $k = 2$ is a measure (easy to see) from the definition. For $k > 2$ is not a measure but a *poly-measure* (i.e. is a measure in each components, but not jointly).

Remark 8. We have that the reduced Schwinger functions $S_{k,\mathbb{A}_k}$ are holomorphic in $\{\mathrm{Re}(t_i) > 0 \colon i = 1, \ldots, k\} \subset \mathbb{C}^k$ and continuous in $\{\mathrm{Re}(t_i) \geqslant 0 \colon i = 1, \ldots, k\}$, moreover we have

$$W_{k,\mathbb{A}_k}(s_1, \ldots, s_{k-1}) = S_{k,\mathbb{A}_k}(i s_1, \ldots, i s_{k-1})$$

where the r.h.s is defined as the limit

$$S_{k,\mathbb{A}_k}(i s_1, \ldots, \mathrm{is}_{k-1}) = \lim_{\lambda_1, \ldots, \lambda_{k-1} \to 0+} S_{k,\mathbb{A}_k}(\lambda_1 + i s_1, \ldots, \lambda_{k-1} + i s_{k-1}).$$

This follows directly from the fact that $S_{k,\mathbb{A}_k}$ is the Laplace transform of a tempered distribution supported on $\mathbb{R}_+^{k-1}$.

Definition 9. *Let $(\tilde{S}_k)_k$ as before. They satisfy Axiom S3 (or reflection positivity) if for $k \in \mathbb{N}$, $j_1, \ldots, j_k \in \mathbb{N}$ and $T_{n-1,j} = (t_{1,(n-1,j)}, \ldots, t_{n-1,(n-1,j)}) \in \mathbb{R}_+^{n-1}$ and $\lambda_{n,j} \in \mathbb{C}$ ($n \leqslant k$ on $j \leqslant j_n$)*

$$\sum_{n_1,n_2=1}^{k} \sum_{h_1=1}^{j_{n_1}} \sum_{h_2=1}^{j_{n_2}} \lambda_{n_1,h_1} \overline{\lambda_{n_2,h_2}} S_{n_1+n_2-1, \theta(\mathbb{A}_{n_2,h_2}) \cdot \mathbb{A}_{n_1,h_1}}(\hat{\theta}(T_{n_2-1,h_2}), T_{n_1-1,h_1}) \geqslant 0$$

This property derives from the fact that the Hilbert scalar product is Hermitian and positive definite, that Q_0 is a representation and that K is self-adjoint (which is linked with the form of $\hat{\theta}$).

Lemma 10. *The Schwinger functions satisfy Axiom S3.*

Now the important result, the reconstruction theorem.

Theorem 11. *Assume that $(\tilde{S}_{k,\mathbb{A}_k})_k$ satisfy Axioms S1,S2,S3. The there exists $(\mathcal{H}, Q_0, (K(t))_t, h_0)$ such that $(\tilde{S}_{k,\mathbb{A}_k})_k$ are the Schwinger functions generated by $(\mathcal{H}, Q_0, (K(t))_t, h_0)$.*

Remark 12. An analogous theorem holds for families $(\tilde{W}_{k,\mathbb{A}_k})_k$ satisfying W1,W2,W3, from which one can construct data $(\mathcal{H}, Q_0, (U(t))_t, h_0)$ for which they are the Wightman functions.

Proof. Let $\mathcal{F}$ be the free algebra generated by the symbols $\tilde{Q}_0(a)$ and $\tilde{K}(t)$ where $a \in \mathcal{A}$ and $t \in \mathbb{R}_+$ equipped with the relations

i. $\tilde{Q}_0(a)\tilde{Q}_0(b) = \tilde{Q}_0(ab)$, $\lambda\tilde{Q}_0(a) + \mu\tilde{Q}_0(b) = \tilde{Q}_0(\lambda a + \mu b)$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$

ii. $\tilde{Q}_0(1_{\mathcal{A}}) = 1_{\mathcal{F}}$

iii. $\tilde{K}(t_1)\tilde{K}(t_2) = \tilde{K}(t_1 + t_2)$

iv. $\tilde{K}(0) = 1_{\mathcal{F}}$

By definition $\mathcal{F}$ is the complex vector space generated by the words of the form $\tilde{Q}_0(a)\tilde{Q}_0(b)\tilde{K}(t)\cdots\tilde{Q}_0(c)\tilde{K}(t')$ which then is extended to an algebra by justapposition of the linear generators and then we take the quotient wrt. the relations listed above. Introduce a useful notation: if $T_{k-1} = (t_1, \ldots, t_{k-1}) \in \mathbb{R}_+^{k-1}$ and $\mathbb{A}_k = (a_1, \ldots, a_k) \in \mathcal{A}^k$, we call $\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \tilde{Q}_0(a_1)\tilde{K}(t_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k) \in \mathcal{F}$. Using the previous relations we have that if $A \in \mathcal{F}$ then

$$A = \sum_{n=1}^{k} \sum_{h=1}^{j_n} \lambda_{n,h} \mathbb{F}_n(T_{n-1,h}, \mathbb{A}_{n-1,h})$$

for some $\lambda_{n,h}, T_{n-1,h}, \mathbb{A}_{n-1,h}$ (in general not in a unique way). On $\mathcal{F}$ we define the scalar product $\langle *, * \rangle_{\mathcal{F}}$ by

$$\langle \mathbb{F}_k(T_{k-1}, \mathbb{A}_k), \mathbb{F}_{k'}(T'_{k'-1}, \mathbb{A}'_k) \rangle_{\mathcal{F}} = \tilde{S}_{k+k'-1, \theta(\mathbb{A}'_{k'})\cdot\mathbb{A}_k}(\hat{\theta}(T'_{k'-1}), T_{k-1})$$

and extend it by linearity to all $\mathcal{F}$ in the first component and by antilinearity in the second component. This definition is well posed since $(\tilde{S}_{k,\mathbb{A}_k})_k$ satisfy the compatibility conditions of Axiom S1 and moreover by the last of the property in Axiom S1 we have that the form $\langle *, * \rangle_{\mathcal{F}}$ is Hermitian and for Axiom S3 that this scalar product is positive definite. We define the linear subspace $\mathcal{N} = \{A \in \mathcal{F}, \langle A, A \rangle_{\mathcal{F}} = 0\}$ and we define $\mathcal{H}_0 = \mathcal{F} \setminus \mathcal{N}$ as a vector space. On $\mathcal{H}_0$ we define $\langle [A], [B] \rangle_{\mathcal{H}} = \langle A, B \rangle_{\mathcal{F}}$ which is well defined by the Cauchy–Schwartz inequality and where $[A] \in \mathcal{H}_0$ denotes the class of $A \in \mathcal{F}$. Moreover we let $\mathcal{H}$ the completion of $\mathcal{H}_0$ with respect to this non-degenerate scalar product $\langle , \rangle_{\mathcal{H}}$ (which is stricly positive on $\mathcal{H}_0 - \{0\}$). We let $h_0 = [1_{\mathcal{F}}]$. We define $\mathbb{K}_t : \mathcal{F} \to \mathcal{F}$ linear such that

$$\mathbb{K}_t(\mathbb{F}_k(T_{k-1}, \mathbb{A}_k)) := \tilde{K}(t)\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathcal{A}}, \mathbb{A}_k)).$$

We have that $\mathbb{K}_t\mathbb{K}_s = \mathbb{K}_{t+s}$ and $\mathbb{K}_0 = 1$. Moreover $\mathbb{K}_t$ is symmetric wrt. the scalar product on $\mathcal{F}$ (this is a consequence of Axiom S1), indeed

$$\langle \mathbb{K}_t(\mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle = \langle \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathcal{A}}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle$$

$$= S_{k+h-1, \theta(\mathbb{A}'_h)(1_{\mathcal{A}}, \mathbb{A}_k)}(\hat{\theta}(T'_{h-1}), (t, T_{k-1})) = S_{k+h-1, \theta((1_{\mathcal{A}}, \mathbb{A}'_h))\mathbb{A}_k}(\hat{\theta}(t, T'_{h-1}), T_{k-1})$$

$$= \langle \mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k), \mathbb{K}_t\mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle$$

and this extends by linearity to deduce the symmetry for $\mathbb{K}_t$. (We continue next week) □