Lecture 23 – 2020.7.7 – 14:15 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Definition 1. *We say that* $\tilde{S}_{k,\cdot}\cdot:\mathscr{A}^k\times\mathbb{R}_+^{k-1}\to\mathbb{C}$ *is linear in* $\mathscr{A}$ *(or satisfies Axiom S0) if for all* $a_1,\ldots,a_k\in\mathscr{A}$ *and* $t_1,\ldots t_{k-1}\in\mathbb{R}_+$ *if the map*

$$a\mapsto\tilde{S}_{k,(a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_k)}(t_1,\ldots,t_{k-1})$$

is linear in $a\in\mathscr{A}$*.*

Recall the theorem we were proving in the last lecture.

Theorem 2. *If* $\tilde{S}_{k,\cdot}\cdot:\mathscr{A}^k\times\mathbb{R}_+^{k-1}\to\mathbb{C}$ *satisfy Axioms S0,S1,S2,S3 we have that there exists an Hilbert space* $\mathscr{H}$*, a representation* Q_0 *of* $\mathscr{A}$ *in* $\mathscr{B}(\mathscr{H})$ *and a self-adjoint, strongly continuous semigroup* $(K(t))_{t\geqslant 0}$ *on* $\mathscr{H}$ *and a vector* $h_0\in\mathscr{H}$ *cyclic wrt.* Q_0,K *and invariant, i.e.* $K(t)h_0=h_0$ *(in other words,* h_0 *is a ground state), such that* $\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})$ *are the Schwinger functions generated by* $(\mathscr{H},Q_0,K,h_0)$*.*

Proof. We introduced the free algebra $\mathscr{F}$ generated by symbols $\tilde{K}(t)$ and $\tilde{Q}_0(a)$ with suitable multiplication rules for $\tilde{Q}_0$ and $\tilde{K}$. We let $\mathbb{F}_k(\mathbb{A}_k,T_{k-1})=\tilde{Q}_0(a_1)\tilde{K}(t_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k)$ and we introduced the positive Hermitian form $\langle\rangle_{\mathscr{F}}$ given on element as

$$\langle\mathbb{F}_k(\mathbb{A}_k,T_{k-1}),\mathbb{F}_h(\mathbb{A}'_h,T'_{h-1})\rangle=S_{k+h-1,\theta(\mathbb{A}'_h)}(\hat{\theta}(T'_{h-1}),T_{k-1}).$$

We have also introduced the null space $\mathscr{N}$ where this form vanishes and $\hat{\mathscr{H}}=\mathscr{F}\setminus\mathscr{N}$ and we let $\mathscr{H}$ the completion of $\mathscr{H}$ wrt. to the scalar product induced by the above form. We introduced also operators $\mathbb{K}(t)$ on $\mathscr{F}$ such that $\mathbb{K}(t)A=\tilde{K}(t)A$ and we proved that it is symmetric wrt. $\langle\rangle_{\mathscr{F}}$ and a semirgoup. We have that

$$\langle\mathbb{K}(t)A,A\rangle=\langle K(t/2)A,K(t/2)A\rangle\geqslant$$

moreover by repeated use of Cauchy–Schwartz we also have

$$\langle\mathbb{K}(t)A,A\rangle\leqslant(\langle\mathbb{K}(2t)A,A\rangle)^{1/2}(\langle A,A\rangle)^{1/2}<\cdots<(\langle\mathbb{K}(2^n t)A,A\rangle)^{1/2^n}(\langle A,A\rangle)^{1-1/2^n}$$

By Axiom S2 we know that $\langle\mathbb{K}(2^n t)A,A\rangle$ can be written as a sum of the form

$$\langle\mathbb{K}(2^n t)A,A\rangle=\sum_{k,\mathbb{A}_k}S_{k,\mathbb{A}_k}(2^n t,t_1,\ldots,t_{k-2})$$

where everything does not depends on n and is uniformly bounded so the quantity $\langle\mathbb{K}(2^n t)A,A\rangle$ is bounded uniformly in n. So

$$\langle\mathbb{K}(t)A,A\rangle\leqslant C^{1/2^n}(\langle A,A\rangle)^{1-1/2^n}$$

and taking $n\to\infty$ we have

$$\langle\mathbb{K}(t)A,A\rangle\leqslant\langle A,A\rangle$$

so $\mathbb{K}(t)\mathscr{N}\subset\mathscr{N}$ and $\mathbb{K}(t)$ is well defined on $\hat{\mathscr{H}}$ and we let $K_0(t)[A]=[\mathbb{K}(t)A]$. We have that for all $t\geqslant 0$

$$\langle K_0(t/2)[A],K_0(t/2)[A]\rangle=\langle K_0(t)[A],[A]\rangle\leqslant\langle[A],[A]\rangle$$

so $K_0(t)$ is a contraction for all $t \geqslant 0$ so it extends to $\mathscr{H}$ as K. It is also self-adjoint and a $(K(t))_{t \geqslant 0}$ is a semigroup. For the strong continuity of the family $(K(t))_{t \geqslant 0}$ we observe that the Schwinger functions are continuous at least when considered as a functions of one of the time variables (fixing all the other parameters). This is enough to prove that $t \mapsto K(t)$ is weakly continuous and then strong continuity follows since it is a contraction.

We define a linear map $\mathbb{Q}(a): \mathscr{F} \to \mathscr{F}$ as $\mathbb{Q}(a)A = \tilde{Q}_0(a)A$. It is a representation of $\mathscr{A}$ on $\mathscr{F}$ (this follows from the relations we imposed on the algebra $\mathscr{F}$). We have that is a $*$-represenation:

$$\langle \mathbb{Q}(a)A, B \rangle_{\mathscr{F}} = \langle A, \mathbb{Q}(a^*)B \rangle_{\mathscr{F}}$$

this can be proved by looking at the definition of the Hermitian form. Moreover on can show $\mathbb{Q}(a)\mathscr{N} \subset \mathscr{N}$ so that we can define the operator on $\hat{\mathscr{H}}$. Define the linear functional on $\mathscr{A}$: $L_A(a) = \langle \mathbb{Q}(a)A, A \rangle_{\mathscr{F}}$. It is positive since

$$L_A(bb^*) = \langle \mathbb{Q}(bb^*)A, A \rangle_{\mathscr{F}} = \langle \mathbb{Q}(b)\mathbb{Q}(b^*)A, A \rangle_{\mathscr{F}} = \langle \mathbb{Q}(b^*)A, \mathbb{Q}(b^*)A \rangle_{\mathscr{F}} \geqslant 0.$$

Therefore it is continuous and its norm on $\mathscr{A}^*$ is given by $L_A(1_{\mathscr{A}}) = \langle A, A \rangle_{\mathscr{F}}$ so if $A \in \mathscr{N}$ then $L_A = 0$. From this, in particular we have $0 = L_A(b^* b) = \langle \mathbb{Q}(b)A, \mathbb{Q}(b)A \rangle_{\mathscr{F}}$ so $Q(b)A \in \mathscr{N}$ for any $b \in \mathscr{A}$. We can then pass to the quotient and define $Q_{00}(a)[A] = [\mathbb{Q}(a)A]$. We have also $\|Q_{00}(a)[A]\|_{\mathscr{F}} \leqslant \|a\|_{\mathscr{A}} \|[A]\|_{\mathscr{F}}$ so Q_{00} is bounded and can be extended to $\mathscr{H}$ as a C^*-homomorphism. We let $h_0 = [1_{\mathscr{F}}]$ and by S1 prove that it is invariant. □

Remark 3. We can replace S2 by S2' which is the property that $\tilde{S}_{k,\cdot} \cdot: \mathscr{A}^k \times \mathbb{R}_+^{k-1} \to \mathbb{C}$ are bounded and continuous in each of the time variables separately. This implies that all together (S0,S1,S2,S3) are equivalent to (S0,S1,S2',S3). (Of course S2 is not equivalent to S2').

An example: the Ornstein–Uhlenbeck process.

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and consider a Gaussian process $X: \mathbb{R} \times \Omega \to \mathbb{R}$, that is such that for all $\xi_1, \ldots, \xi_k \in \mathbb{R}$ we have that $(X_{\xi_1}, \ldots, X_{\xi_k})$ is a k-dimensional Gaussian. A Gaussian process is characterised by its mean and covariance function. We let $\mathbb{E}[X_\xi] = 0$ for all $\xi \in \mathbb{R}$ and

$$\mathrm{Cov}(X_\xi, X_{\xi'}) = \mathbb{E}[X_\xi X_{\xi'}] = \frac{1}{2\theta} e^{-\theta|\xi - \xi'|}, \qquad \xi, \xi' \in \mathbb{R}.$$

If $\tilde{S}_{k,\cdot} \cdot: \mathscr{A}^k \times \mathbb{R}_+^{k-1} \to \mathbb{C}$ then we define extended functions $\tilde{\mathscr{S}}_{k,\cdot} \cdot: \mathscr{A}^k \times \mathbb{R}_+^k \to \mathbb{C}$ such that, if $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_k$ we let

$$\tilde{\mathscr{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{S}_{k,\mathbb{A}_k}(\xi_2 - \xi_1, \xi_3 - \xi_2, \ldots, \xi_k - \xi_{k-1})$$

and if $\xi_1, \ldots, \xi_k$ are general then we let

$$\tilde{\mathscr{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{\mathscr{S}}_{k,\mathbb{A}_k}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)})$$

where $\sigma \in S_n$ is the permutation such that $\xi_{\sigma(1)} < \cdots < \xi_{\sigma(k)}$. Note that to a family $\tilde{\mathscr{S}}$ invariant under translation and permutation of the time variables it associated a unique family $\tilde{S}$ and viceversa.

We choose now $\mathscr{A} = C_b^0(\mathbb{R})$ and let

$$\tilde{\mathscr{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \mathbb{E}[a_1(X_{\xi_1}) \cdots a_k(X_{\xi_k})]$$

when $\xi_1 \leqslant \cdots \leqslant \xi_k$ and the extended via permuations as above. This is a symmetric function which is invariant under translation of the time variables, so we can identify the functions $\tilde{S}$ and have

$$\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) = \mathbb{E}[a_1(X_0)a_2(X_{t_1})a_3(X_{t_1+t_2})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

$$= \mathbb{E}[a_1(X_t)a_2(X_{t+t_1})a_3(X_{t+t_1+t_2})\cdots a_k(X_{t+t_1+\cdots+t_{k-1}})]$$

By construction S0,S1 are true and depend only on the linearity of expectations. For S1 we observe that, for example,

$$S_{2,(a_1,a_2)}(0) = \mathbb{E}[a_1(X_0)a_2(X_0)] = \mathbb{E}[(a_1a_2)(X_0)] = S_{1,(a_1a_2)}$$

and similarly for all the other conditions of S1. S2' is true, as easily seen from the definition thanks to convergence in law to prove continuity observing that

$$(X_{\xi_1},\ldots,X_{\xi_{i-1}},X_{\xi},X_{\xi_{i+1}},\ldots,X_{\xi_k}) \underset{\text{law}}{\longrightarrow} (X_{\xi_1},\ldots,X_{\xi_{i-1}},X_{\xi_i},X_{\xi_{i+1}},\ldots,X_{\xi_k})$$

if $\xi \to \xi_i$ since the covariance function is continuous in each variable and the characteristic functions converge (by Lévy's theorem this implies convergence in law), and using

$$|\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})| \leqslant \|a_1\|\cdots\|a_k\|,$$

for the boundedness.

Consider now:

$$S_{k+h-1,\theta(\mathbb{A}'_h)\mathbb{A}_k}(\hat{\theta}(T'_{h-1}),T_{k-1})$$

$$= \mathbb{E}[a'^*_h(X_{-t'_{h-1}-t'_{h-2}-\cdots-t'_1})\cdots a'^*_2(X_{-t_1})a'^*_1(X_0)a_1(X_0)a_2(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

Consider also the transformation $\mathbb{R}$ of the process X defined as $\mathbb{R}(X)_t = X_{-t}$. Then

$$S_{k+h-1,\theta(\mathbb{A}'_h)\mathbb{A}_k}(\hat{\theta}(T'_{h-1}),T_{k-1}) = \mathbb{E}[a'^*_h(\mathbb{R}(X)_{t'_1+\cdots+t'_{h-1}})\cdots a'^*_1(\mathbb{R}(X)_0)a_1(X_0)a_2(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

We denote $F \in \mathscr{C}^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+},\mathbb{C})$ if F is a cylindric continuous function, i.e. if there exists k and $\xi_1,\ldots,\xi_k \in \mathbb{R}_+$ such that there exists unique continuous $\tilde{F}: \mathbb{R}^k \to \mathbb{C}$ such that $F(X) = \tilde{F}(X_{\xi_1},\ldots,X_{\xi_k})$.

Theorem 4. *$(\tilde{S}_k)_k$ satisfy Axiom S3 iff for any $F \in \mathscr{C}^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+},\mathbb{C})$ we have that*

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}X)}] \geqslant 0.$$

Proof. Only a sketch. The implication $\Leftarrow$ is the most important for us. Consider

$$F(X) = \sum_{n=1}^{k}\sum_{h=1}^{j_n} \lambda_{n,j}a_{1,n,j}(X_0)a_{2,n,j}(X_{t_1})\ldots a_{n,n,h}(X_{t_{1,n,h}+\cdots+t_{n,n,h}}).$$

Note now that

$$\sum_{n_1,n_2=1}^{k}\sum_{h_1,h_2=1}^{j_{n_1},j_{n_2}} \lambda_{n_1,h_1}\bar{\lambda}_{n_2,h_2}S_{n_1+n_2-1,\theta(\mathbb{A}_{n_2,h_2})\mathbb{A}_{n_1,h_1}}(\hat{\theta}(t_{1,n_1,h1}\ldots t_{1,n_1,h1})\cdots) = \mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] \geqslant 0.$$

by hypothesis. The converse is also true because the functions of the form $a_1(x_1)\cdots a_k(x_k)$ are dense in $C^0_\infty(\mathbb{R}^k)$ and this last algebra is dense in $C^0_b(\mathbb{R}^k)$ wrt. the pointwise convergence with uniform bounds. □

Theorem 5. *Let $0 \leqslant \eta_1 \leqslant \cdots \leqslant \eta_h$ and $0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_k$. Then $(X_{-\eta_1}, \ldots, X_{-\eta_n})$ is conditionally independent of $(X_{\xi_1,\ldots,\xi_k})$ given X_0.*

Proof. This can be proven... (tomorrow) □