Lecture 24 – 2020.7.8 – 8:30 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

In the last lecture we started to see a method to construct Schwinger functions starting from a stochastic process. We took the Ornstein–Uhlenbeck process, i.e. the real (one-dimensional) Gaussian process on $\mathbb{R}$ with meaan zero and covariance

$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}[X_t X_s] = R(t-s) = \frac{1}{2\theta} e^{-\theta|t-s|}, \qquad t, s \in \mathbb{R}$$

for a given parameter $\theta > 0$. We took $\mathscr{A} = C_b^0(\mathbb{R})$ and we took a candidate extended Schwinger function of the form

$$\tilde{\mathscr{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \mathbb{E}[a_1(X_{\xi_1}) \cdots a_k(X_{\xi_k})]$$

where $\xi_1 \leqslant \cdots \leqslant \xi_k$ and then extended symmetrically to all $\mathbb{R}^k$. We proved that the functions $(\tilde{\mathscr{S}}_{k,\cdot})_k$ satisfy Axioms S0,S1,S2. We also introduced the notation $F \in C^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ for cylindric function, i.e. functions such that there exists $k \geqslant 0$ and $\tilde{F}: \mathbb{R}^k \to \mathbb{C}$ and $\xi_1, \ldots, \xi_k$ such that $F(X) = \tilde{F}(X_{\xi_1}, \ldots, X_{\xi_k})$. We introduced also $\mathbb{R}(X)_t = X_{-t}$. We proved the following equivalent characterisation of S3:

Theorem 1. *$(\tilde{\mathscr{S}}_{k,\cdot})_k$ satisfies S3 iff for all $F \in C^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ we have*

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] \geqslant 0.$$

Definition 2. *A process $\tilde{X}$ such that for all $F \in C^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ we have $\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] \geqslant 0$ it is called a reflection positive process.*

Lemma 3. *Consider (Y_1, Y_2) taking values in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ which are Gaussian random variables with covariance*

$$\mathrm{Cov}(Y) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with $B_{i,j} = \mathrm{Cov}(Y_i, Y_j)$. Then Y_1 given Y_2 is a Gaussian random variable and the conditional covariance is given by

$$\mathrm{Cov}(\mathbb{E}(Y_1|Y_2)) = B_{11} - B_{12} B_{22}^{-1} B_{21}$$

we are assuming that B_{22} is non-singular.

Proof. Exercise. □

Lemma 4. *If $\eta_1, \ldots, \eta_h \geqslant 0$ and $\xi_1, \ldots, \xi_k \geqslant 0$ then $Y_1 = (X_{-\eta_1}, \ldots, X_{-\eta_h})$ is conditionally independent of $Y_2 = (X_{\xi_1}, \ldots, X_{\xi_k})$ given X_0, where X is the OU process above.*

Proof. We have by simple inspection

$$\mathrm{Cov}((Y_1, Y_2, X_0)) = \begin{pmatrix} C_1 & B_1^T & D_1^T \\ B_1 & C_2 & D_2^T \\ D_1 & D_2 & 1/2\theta \end{pmatrix}$$

with

$$D_1 = \left(\frac{e^{-\theta\eta_1}}{2\theta}, \ldots, \frac{e^{-\theta\eta_h}}{2\theta}\right), \quad D_2 = \left(\frac{e^{-\theta\xi_1}}{2\theta}, \ldots, \frac{e^{-\theta\xi_k}}{2\theta}\right)$$

and

$$(B_1)_{i,j} = \frac{e^{-\theta(\eta_i+\xi_j)}}{2\theta}.$$

So

$$\mathrm{Cov}((Y_1, Y_2)|X_0) = \begin{pmatrix} C_1 & B_1^T \\ B_1 & C_2 \end{pmatrix} - (2\theta)(D_1, D_2)^T(D_1, D_2)$$

with

$$(D_1, D_2)^T(D_1, D_2) = \begin{pmatrix} \tilde{D}_1 & B_1^T \\ B_1 & \tilde{D}_2 \end{pmatrix}$$

so finally one has

$$\mathrm{Cov}((Y_1, Y_2)|X_0) = \begin{pmatrix} \tilde{C}_1 & 0 \\ 0 & \tilde{C}_2 \end{pmatrix}$$

for some matrices $\tilde{C}_1, \tilde{C}_2$. The important observation is that the antidiagonal is zero. (check as exercise). From this form of the covariance this implies that Y_1, Y_2 are independent given X_0. □

We are going now to prove

Theorem 5. *The OU process X is reflection positive.*

Proof. Take $F \in C^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$, so $F = \tilde{F}(X_{\xi_1}, \ldots, X_{\xi_k})$ with $\xi_1, \ldots, \xi_k \geqslant 0$ as above. By the conditional independence (annd the complex-linearity of the expectation) we have

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] = \mathbb{E}[\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}|X_0]]$$

$$= \mathbb{E}[\mathbb{E}[F(X)|X_0]\overline{\mathbb{E}[F(\mathbb{R}(X))|X_0]}]$$

Now we observe that X is invariant wrt. reflections so

$$\mathbb{E}[F(\mathbb{R}(X))|X_0] = \mathbb{E}[F(\mathbb{R}(X))|\mathbb{R}(X_0)] = \mathbb{E}[F(X)|X_0]$$

and we obtain

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] = \mathbb{E}[|\mathbb{E}[F(X)|X_0]|^2] \geqslant 0.$$ □

As a consequence we obtain that $(\tilde{\mathscr{S}}_k)_k$ satisfy axioms S0,S1,S2,S3 and by the reconstruction theorem there exists $(\mathscr{H}, Q_0, (K(t))_{t\geqslant 0}, h_0)$ such that $(\tilde{\mathscr{S}}_k)_k$ are the associated extended Schwinger functions.

Now we are interested in explicitly describing these objects in this particular situation.

In this case we can prove that the free algebra $\mathscr{F}$ introduced in the reconstruction is isomorphic to the algebra $\mathscr{F}_X \subseteq C^0_c(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ by identifying

$$\tilde{Q}_0(a_0)\tilde{K}(t_1)\tilde{Q}_0(a_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k)$$

with

$$a_0(X_0)a_1(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})$$

and extending this map by linearity. We leave as an exercise to prove the isomorphism (as algebras). Under this ispomorphis if $F, G \in \mathscr{F}_X$ then we also have that the Hermitian form $\langle\rangle_{\mathscr{F}}$ can be represented probabilistically as

$$\langle F, G\rangle_{\mathscr{F}_X} = \mathbb{E}[F(X)\overline{G(\mathbb{R}(X))}]$$

which we know to be non-negative and Hermitian. Let $\mathscr{N}_X := \{F \in \mathscr{F}_X | \langle F, F\rangle_{\mathscr{F}_X} = 0\} \subseteq \mathscr{F}_X$

Remark 6. If $F \in \mathscr{F}_X$ then there exists a version $\mathbb{E}[F|X_0]$ which belongs to $\mathscr{F}_X$, indeed the conditional expectation can be written as $\mathbb{E}[F|X_0] = F(L_F X_0)$ for soem linear map L_F depending on F

Lemma 7. *We have*

$$F - \mathbb{E}[F|X_0] \in \mathscr{N}_X$$

Proof. Observe that

$$\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0][\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]] = 0$$

since clearly $\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0] = 0$. □

So from an algebraic point of view we have that $\hat{\mathscr{H}} = \mathscr{F}_X \backslash \mathscr{N}_X$ is just $C_b^0(\mathbb{R}, \mathbb{C})$ where the map $\mathscr{F}_X \to \hat{\mathscr{H}}$ is just the conditional expectation $F \mapsto \mathbb{E}[F|X_0]$. That $\hat{\mathscr{H}} = C_b^0(\mathbb{R}, \mathbb{C})$ is clear since $\mathbb{E}[a_0(X_0)|X_0] = a_0(X_0)$ so it is a surjective mapping. Moreover the scalar product can be written

$$\langle f, g\rangle_{\hat{\mathscr{H}}} = \mathbb{E}[f(X_0)\overline{g(X_0)}] = \int_{\mathbb{R}} f(z)\overline{g(z)} \underbrace{\frac{e^{-\theta z^2/2}}{(2\pi/\theta)^{1/2}} \mathrm{d}z}_{\mu_\theta(\mathrm{d}z)}$$

and as a consequence $\mathscr{H} = L^2(\mathbb{R}, \mathbb{C}, \mu_\theta)$ moreover $(Q_0(a)f)(z) = a(z)f(z)$. Recall now that $\mathbb{K}(t)F = \tilde{K}(t)A$ which under our isomorphism it is send to a translation of the time variable:

$$\mathbb{K}(t)F(X) = F(X_{t+\cdot}).$$

In particular $\mathbb{K}(t)f(X_0) = f(X_t)$ and we have

$$(K(t)f)(X_0) = \mathbb{E}[\mathbb{K}(t)f(X_0)|X_0] = \mathbb{E}[f(X_t)|X_0]$$

This conditional expectation can be written explicitly since $\mathrm{Cov}(X_t, X_0) = (2\theta)e^{-\theta t}$ and so

$$X_t = e^{-\theta t}X_0 + (1 - e^{-2\theta t})^{1/2} N_\theta$$

where $N_\theta \sim \mathscr{N}(0, 1/2\theta)$ and it is independent of X_0, then

$$K(t)f(z) = \mathbb{E}[f(X_t)|X_0 = z] = \mathbb{E}[f(e^{-\theta t}z + (1 - e^{-2\theta t})^{1/2} N_\theta)] = \int_{\mathbb{R}} f(e^{-\theta t}z + (1 - e^{-2\theta t})^{1/2} y)\mu_\theta(\mathrm{d}y).$$

Obviously $h_0 = 1 \in L^2(\mathbb{R}, \mathbb{C}, \mu_\theta)$. From the explicit expression of $(K(t))_{t\geqslant 0}$ one can check again that it is a stronly continuous contraction semigroup. This is called the Ornstein–Uhlenbeck semigroup.

This is not what is done usually in quantum mechanics since the usual space there is taken to be $L^2(\mathbb{R}, \lambda)$ where λ is the Lebesgue measure, not μ_θ. The map connecting the two representations is

$$f \in \hat{\mathscr{H}} \to \tilde{f}(z) = f(z)\frac{e^{-\theta z^2/4}}{(2\pi/\theta)^{1/4}} \in \tilde{\mathscr{H}} = L^2(\mathbb{R}, \lambda)$$

Let's compute the generator H of $K(t)$:

$$-Hf(z) = \lim_{t\to 0}\frac{K(t)f(z) - f(z)}{t} = \lim_{t\to 0}\int_{\mathbb{R}}\frac{f(e^{-\theta t}z + (1-e^{-2\theta t})^{1/2}y) - f(z)}{t}\mu_\theta(\mathrm{d}y)$$

By Taylor expansion:

$$= \lim_{t\to 0}\int_{\mathbb{R}}\frac{f'(z)((e^{-\theta t}-1)z + (1-e^{-2\theta t})^{1/2}y) + \frac{1}{2}f''(z)((e^{-\theta t}-1)z + (1-e^{-2\theta t})^{1/2}y)^2 + O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

and since μ_θ has zero first moment we have

$$= \lim_{t\to 0}\int_{\mathbb{R}}\frac{f'(z)(e^{-\theta t}-1)z + \frac{1}{2}f''(z)((1-e^{-2\theta t})^{1/2}y)^2 + O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

$$= \lim_{t\to 0}\frac{f'(z)(-\theta t)z + \frac{1}{2}f''(z)(1-e^{-2\theta t})(1/2\theta) + O(t^{3/2})}{t} = -\theta f'(z) + \frac{1}{4}f''(z)$$

so on $\hat{\mathscr{H}}$ we have

$$Hf(z) = \theta f'(z) - \frac{1}{4}f''(z)$$

and the same operator on $\tilde{\mathscr{H}}$ has the form

$$\tilde{H}f(z) = -\theta z^2 \tilde{f}(z) - \frac{1}{4}\Delta\tilde{f}(z)$$

and this is usually called the Schrödinger representation of the harmonic oscillator, indeed note that

$$\tilde{H} = \frac{1}{4}P^2 + Q^2\frac{\theta^2}{2}$$

which if interpreted classically is the Hamiltonian of the harmonic oscillator.

Therefore we have proven that the quantum mechanical harmonic oscillator is related via the reconstruction theorem with the Ornstein–Uhlenbeck process.

Next time we will take a state space M and a stochastic process $(X_t)_{t\in\mathbb{R}}$ taking values in M and take $\mathscr{A}$ a subset of the continuous functions on M large enough (so that $\mathscr{A}$ characterise the measures on M) and we defined the Schwinger functions as before, i.e. as

$$\mathscr{S}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \mathbb{E}[a_1(X_{\xi_1})\cdots a_k(X_{\xi_k})]$$

and the properties $S0, S1, S2, S3$ become suitable probabilistic properties of $(X_t)_{t\in\mathbb{R}}$. We will see what are exaclty there probabilistic properties (invariance, continuity in disitrbuion, and reflection positivity). We are then going to characterise some classes of processes which have these properties (and therefore which give rise to quantum mechanical dynamics).

Why it is simpler to use this strategy (to construct QM models): essentially because probabilistic tools are usually easier to use/more powerful than functional analityc tools in Hilbert spaces. So the probabilistic model should be considered a special and versatile representation of a quantum system.