Lecture 25 – 2020.7.14 – 14:15 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

In the last lecture we introduced four axioms for functions

$$\mathcal{S}_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_k)=S_{k,\mathbb{A}_k}(\xi_2-\xi_1,\ldots,\xi_k-\xi_{k-1})=S_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})$$

Axioms:

S0. Linearity in $a\in\mathcal{A}$

S1. Compatibility conditions

S2. Laplace transform of a positively supported distribution

S2'. Boundedness and continuity in t

S3. Reflection positivity.

We have proven that

$$(S0,S1,S2,S3)\Leftrightarrow(S0,S1,S2',S3)$$

and that they are equivalent to the existence of the data of a quantum system with ground state (I do not repreat here the formulation).

Last week we introduced a stochastic process and showed that it can be used to define Schwinger functions satisfying the above Axioms.

We want now to generalise this setting. We consider now $\mathcal{A}\subset C_b^0(M)$ where M is topological space.

We introduce now Axiom N (Nelson positivity).

Definition 1. *A family $(S_k)_k$ is Nelson positive is for all $t_1,\ldots,t_{k-1}\in\mathbb{R}_+$ there exists $\mu_{t_1,\ldots,t_{k-1}}$ a Radon probability measure on M^k such that*

$$S_{k,(a_1,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\int_{M^k}a_1(x_1)\cdots a_k(x_k)\,\mu_{t_1,\ldots,t_{k-1}}(\mathrm{d}x_1\cdots\mathrm{d}x_k)$$

Remark 2. In particular, if $a_1,\ldots,a_k\geqslant 0$ in $\mathcal{A}$ i.e. $a_i=b_ib_i^*$ then

$$S_{k,(a_1,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\int_{M^k}|b_1(x_1)\cdots b_k(x_k)|^2\mu_{t_1,\ldots,t_{k-1}}(\mathrm{d}x_1\cdots\mathrm{d}x_k)\geqslant 0.$$

This justify the name of positivity.

On M we need to assume also that

(*). $\mathcal{A}^{\otimes k}$ (the linear combination of functions of the form $a_1(x_1)\cdots a_k(x_k)$) generates $C_b^0(M^k;\mathbb{C})$ with respect to the topology of pointwise convergence with uniform bounds.

For example, this holds, if $M=\mathbb{R}^m$ and $\mathcal{A}$ is the space of continuous functions vanishing at ∞ on M,

Theorem 3. *$(S_k)_k$ satisfy Axioms (N, S1, S2, S3) is equivalent to the existence of a stochastic process X: $\Omega \times \mathbb{R} \to M$ such that*

1. $$\mathscr{S}_{k, \mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \mathbb{E}[a_1(X_{\xi_1}) \cdots a_k(X_{\xi_k})]$$

2. *$(X_{\xi_1}, \ldots, X_{\xi_i}, \ldots, X_{\xi_k}) \to (X_{\xi_1}, \ldots, X_{\xi}, \ldots, X_{\xi_k})$ in law as $\xi_i \to \xi \in \mathbb{R}$.*

3. *For any $s \in \mathbb{R}$ we have that $(X_{s+t})_{t \in \mathbb{R}}$ has the same law of X, i.e. the law of X is invariant under translation*

4. *Recall that $\mathbb{R}(X)_t = X_{-t}$ and that $F \in C^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+}; \mathbb{C})$ with $F(X) = \tilde{F}(X_{\xi_1}, \ldots, X_{\xi_k})$, then we have that*

$$\mathbb{E}[F(X) F(\mathbb{R}(X))] \geqslant 0,$$

i.e. the process X is reflection positive.

Proof. The direction $\Leftarrow$ is the same in the case $M = \mathbb{R}$ and X the OU process, we did in the last lectures. The reverse direction $\Rightarrow$ goes as follows. If there exists a process satisfying condition 1 using the technical hypothesis $(*)$ we can prove 2,3,4. Indeed if $\mathscr{S}$ satisfies Axiom S0,S1,S2,S3 the process X satisfies 4 for $F = \sum_m \lambda_m a_{1,m}(x_1) \cdots a_{k,m}(x_k)$ but by $(*)$ the functions of this form are dense in $C^0_{\text{cyl}}(M^{\mathbb{R}_+}, \mathbb{C})$ with respect to the pointwise convergence with uniform bounds so 4 follows from dominated convergence theorem. For 2 we do the the case involving only one function:

$$\mathscr{S}_{1,(a_1)}(\xi_1) = \mathbb{E}[a_1(X_{\xi_1})]$$

but $S2'$ implies $\lim_{\xi_1 \to \xi} \mathscr{S}_{1,(a_1)}(\xi_1) = \mathscr{S}_{1,(a_1)}(\xi) = \mathbb{E}[a_1(X_{\xi})]$ but they are dense in $C^0_b(M, \mathbb{C})$ and one can argument the convergence in law. For 3 one uses the fact that the function are invariant under translations and $(*)$. It remains now to prove 1, i.e. the existence of such a process. By N we have that

$$S_{k,(a_1, \ldots, a_k)}(t_1, \ldots, t_{k-1}) = \int_{M^k} a_1(y_1) \cdots a_k(y_k) \, \mu_{t_1, \ldots, t_{k-1}}(\mathrm{d}y_1 \cdots \mathrm{d}y_k)$$

for some Radon probability measure $\mu_{t_1, \ldots, t_{k-1}}$. We consider the process $(X_\xi)_\xi$ with marginals given by $\mu_{t_1, \ldots, t_{k-1}}$. The law of X is unique (if exists) because of $(*)$. By Axiom $S1$ (compatibility conditions), in particular the fact that $\mathscr{S}_{k,(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_k)}(\xi_1, \ldots, \xi_k) = \mathscr{S}_{k-1,(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_k)}(\xi_1, \ldots, \cancel{\xi_i}, \ldots, \xi_k)$ and this implies that $(\mu_{T_k})_{T_k}$ are a compatible family of finite dimensional marginals, and by Kolmogorov's extension theorem there exists a probability measure $\mathbb{P}$ on $\Omega = M^{\mathbb{R}}$ witth the product σ-algebra and with marginals given by $\mu_{\xi_1, \ldots, \xi_k}$. So we can take on Ω the process $X: M^{\mathbb{R}} \times \mathbb{R} \to M$ given by $X(\omega)(t) = \omega(t)$. □

The most difficult of the conditions is the reflection positivity. There is no "easy" way to check for it, however is a quite robust property which pass easily to the limit. In this second property it lies its usefulness.

Situations in which one can check easily for reflection positivity are two. The first is when dealing with Gaussian processes, then second in when dealing with Markov processes.

We focus today on the Gaussian case. Let $M = \mathbb{R}^m$ and $\mathscr{A} = C^0_b(M; \mathbb{C})$ and X_t a Gaussian process taking values in $\mathbb{R}^m$ with mean zero. For $\alpha \in \mathbb{R}^m$ we can define $\alpha \cdot X_t = \sum \alpha_i X^i_t$. A Gaussian process is uniquely characterised by its covariance function

$$r^{ij}(t, s) = \mathbb{E}[X^i_t X^j_s].$$

If X satisfies condition 3 then we have that $r^{ij}(t, s)$ is only a function of $t - s$, i.e. $r^{ij}(t, s) = r^{ij}(t - s)$. The continuity in distribution is equivalent to require that $t \mapsto r^{i,j}(t)$ is continuous. This can be verified using the characteristic function (exercise). What about reflection positivity?

Theorem 4. *If X is a reflection positive process then for all $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $\xi_1, \ldots, \xi_k \in \mathbb{R}$ we have*

$$\sum_{i,j=1}^{k} \langle \alpha_i, r(\xi_i + \xi_j) \bar{\alpha}_j \rangle_{\mathbb{R}^m} \geqslant 0. \tag{1}$$

Proof. We prove in the scalar case $m = 1$ and $M = \mathbb{R}$, the general case follows similarly. We consider $f_n \in \mathscr{A} \to x$ in $\mathbb{R}$ and such that $|f(x)| \leqslant |x|$, e.g. $f_n(x) = (-n) \vee (x \wedge n)$. Let $F_n(x) = \sum_i \alpha_i f_n(X_{\xi_i})$, then

$$0 \leqslant \mathbb{E}[F_n(X) \overline{F_n(\mathbb{R}(X))}] = \sum_{i,j=1}^{k} \alpha_i \bar{\alpha}_j \mathbb{E}[f_n(X_{\xi_i}) f_n(X_{-\xi_j})] \to \sum_{i,j=1}^{k} \langle \alpha_i, r(\xi_i + \xi_j) \bar{\alpha}_j \rangle_{\mathbb{R}^m}$$

by Lebesgue dominated convergence theorem. □

Theorem 5. *(Wick's theorem) Let $(Y_1, \ldots, Y_k)$ be a centred Gaussian vector, then for r even and $i_1, \ldots, i_r$ chosen among $\{1, \ldots, k\}$ we have*

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_r}] = \sum_{\{(i,j)\}} \prod_{(i,j) \in \{(i,j)\}} \mathbb{E}[Y_i Y_j]$$

where $\{(i,j)\}$ run over the perfect matches of $\{i_1, \ldots, i_r\}$. If r is odd then the expectation is zero.

Proof. Let $\Sigma_{i,j} = \mathbb{E}[Y_i, Y_j]$ and we have that the moment generating function is given by

$$\mathbb{E}[e^{\alpha \cdot Y}] = e^{\frac{1}{2}\langle \alpha, \Sigma \alpha \rangle}$$

then

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_r}] = \frac{\partial^r}{\partial \alpha_{i_1} \cdots \partial \alpha_{i_r}} \bigg|_{\alpha = 0} \mathbb{E}[e^{\alpha \cdot Y}] = \cdots = \sum_{\{(i,j)\}} \prod_{(i,j) \in \{(i,j)\}} \mathbb{E}[Y_i Y_j].$$

□

Lemma 6. *Let $(Y_1, \ldots, Y_k)$ be Gaussian with mean zero, then there are polynomials $p_N(x) \in C^0(\mathbb{R}^k, \mathbb{R})$ indexed by $N = \{i_1, \ldots, i_r\}$ with r even or odd of the form*

$$p_N(x) = x_{i_1} \cdots x_{i_r} - \sum_{M : M < N} c_M p_M(x)$$

where M is of degree less then N. These polynomials are orthogonal wrt. the Gaussian measure, i.e.

$$\mathbb{E}[p_N(Y) p_M(Y)] = 0$$

for $\deg(N) \neq \deg(M)$.

Proof. If the covariance matrix Σ is non-singular we apply a form Gram–Schmidt orthogonalisation. For Σ general we can find a subset of the Gaussians whose covariance is non-singular and express the rest of the random variables by linear combinations of this subset and use the previous method. □

Theorem 7. *Given eq.(1) + 1. + 2. then X is reflection positive.*

Proof. (to be done tomorrow) □