Lecture 26 – 2020.7.15 – 8:30 via Zoom – F. de Vecchi

(Script by M. Gubinelli of the lecture of Francesco.)

Note that in the last lecture we had

$$r(t-s)=\mathbb{E}[X_t X_s]=r(s-t)$$

so in particular $r(t)=r(|t|)$.

The lemma on orthogonal polinomials holds actually for any random variable (for which polinomials are integrable). In the Gaussian case we can prove that the polynomial depends only on the variables we are considering. Let us give here the version of the lemma that we are going to actually use.

Lemma 1. *Let $(Y_1,\ldots,Y_k)$ in $\mathbb{R}^k$ be Gaussian random variables. Use $N=\{i_1,\ldots,i_r\}$ for multiindices. There exists polynomials $p_N(y_{i_1},\ldots,y_{i_r})$ such that*

$$p_N(y_{i_1},\ldots,y_{i_r})=y_{i_1}\cdots y_{i_r}+\textit{lower order polynomial}.$$

and

$$\mathbb{E}[p_N(Y_{i_1},\ldots,Y_{i_r})p_{N'}(Y_{i'_1},\ldots,Y_{i'_{r'}})]=0$$

if $r\neq r'$. Moreover introducing the notion of Wick product we have: the Wick product is $:Y_{i_1}\cdots Y_{i_r}:=p_N(Y_{i_1},\ldots,Y_{i_r})$ which is characterised by the properties

$$\frac{\partial}{\partial Y_{i_j}}:Y_{i_1}\cdots Y_{i_r}:=:Y_{i_1}\cdots \not{Y}_{i_j}\cdots Y_{i_r}:,\qquad \mathbb{E}[:Y_{i_1}\cdots Y_{i_r}:]=0.$$

Note that $:Y_i:=Y_i$.

Proof. The proof is based on Wick's theorem. If Q_1,Q_2 are two polynomials $Q_1(Y_{i_1},\ldots,Y_{i_r})$ and $Q_2(Y_{j_1},\ldots,Y_{j_\ell})$ then

$$\mathbb{E}[Q_1(Y_{i_1},\ldots,Y_{i_r})Q_2(Y_{j_1},\ldots,Y_{j_\ell})]=\sum_{p,q}\mathbb{E}[Y_{i_p}Y_{j_q}]\mathbb{E}\left[\left(\frac{\partial}{\partial Y_{i_p}}Q_1(Y_{i_1},\ldots,Y_{i_r})\right)\left(\frac{\partial}{\partial Y_{j_q}}Q_2(Y_{j_1},\ldots,Y_{j_\ell})\right)\right]$$

which can be proven by integration by parts on monomials and then extended by linearity. We want to prove now that

$$\mathbb{E}[:Y_{i_1}\cdots Y_{i_r}::Y_{j_1}\cdots Y_{j_\ell}:]=0$$

for $r\neq\ell$. The proof is by induction on $r+\ell$, when $r+\ell=1$ we have $\mathbb{E}[:Y_i:]=\mathbb{E}[Y_i]=0$. Otherwise we use the above formula to have

$$\mathbb{E}[:Y_{i_1}\cdots Y_{i_r}::Y_{j_1}\cdots Y_{j_\ell}:]=\sum_{p,q}\mathbb{E}[Y_{i_p}Y_{j_q}]\mathbb{E}\left[:Y_{i_1}\cdots \not{Y}_{i_p}\cdots Y_{i_r}::Y_{j_1}\cdots \not{Y}_{j_q}\cdots Y_{j_\ell}:\right]=0$$

using the induction hypothesis. □

Theorem 2. *Assume that the covariance r satisfies*

$$\sum_{i,j=1}^{k}\langle\alpha_i, r(t_i+t_j)\bar{\alpha}_j\rangle_{\mathbb{R}^m}\geqslant 0. \tag{1}$$

for all $\alpha_i \in \mathbb{C}$ and $t_i \in \mathbb{R}$. Then X is a reflection positive process.

Proof. The first step is to prove that reflection positivity holds for polynomials and then extended to arbitrary functions. Take a cylindrical polynomial $Q(X) = \tilde{Q}(X_{\xi_1},\ldots,X_{\xi_k})$ for some $k \geqslant 1$ and $\xi_1,\ldots,\xi_k \in \mathbb{R}$. This polynomial can be expanded in Wick products (since they span the space of all polynomials). We consier the scalar case, the vector case just involve heavier notation. We have

$$\tilde{Q}(X_{\xi_1},\ldots,X_{\xi_k}) = \sum \lambda_{i_1,\ldots,i_r} {:}X_{\xi_{i_1}}\cdots X_{\xi_{i_r}}{:}$$

with $\lambda_{i_1,\ldots,i_r} \in \mathbb{C}$. Note that if we let ${:}X_{\xi_{i_1}}\cdots X_{\xi_{i_r}}{:} = f(X)$ then ${:}X_{-\xi_{i_1}}\cdots X_{-\xi_{i_r}}{:} = f(\mathbb{R}(X))$ since the covariance is invariant under reflections. Then

$$\mathbb{E}[Q(X)\overline{Q(\mathbb{R}(X))}] = \sum \lambda_{i_1,\ldots,i_r} \overline{\lambda_{j_1,\ldots,j_\ell}} \mathbb{E}[{:}X_{\xi_{i_1}}\cdots X_{\xi_{i_r}}{::}X_{-\xi_{j_1}}\cdots X_{-\xi_{j_\ell}}{:}]$$

$$= \sum \lambda_{i_1,\ldots,i_r} \overline{\lambda_{j_1,\ldots,j_\ell}} \sum_{\text{pairings } (i_q,j_p)} \prod r(\xi_{i_q} + \xi_{j_p})$$

where we use that if $r = \ell$ we have

$$\mathbb{E}[{:}X_{\xi_{i_1}}\cdots X_{\xi_{i_r}}{::}X_{\xi_{j_1}}\cdots X_{\xi_{j_\ell}}{:}] = \sum_{q,p} \mathbb{E}[X_{\xi_q} X_{-\xi_p}] \mathbb{E}\Big[{:}X_{\xi_{i_1}}\cdots \not{X}_{\xi_q}\cdots X_{\xi_{i_r}}{::}X_{\xi_{j_1}}\cdots \not{X}_{\xi_p}\cdots X_{\xi_{j_\ell}}{:}\Big]$$

and proceeding with this we obtain the equality above. We have now to show that the above expression is positive, we know that the matrix $(r(\xi_{i_q} + \xi_{j_p}))_{p,q}$ is positive definite and so the above expression can be written as $\langle v_1, v_2 \rangle_{\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}}$ where on the vector space $\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}$ we consider the scalar products where on $\mathbb{R}^k$ we consider the product

$$\sum_i \alpha_i \bar{\alpha}_j r(\xi_i + \xi_j)$$

while on $(\mathbb{R}^k)^{\otimes \ell}$ we use the tesorization of this scalar product, i.e. for $p_1 \otimes \cdots \otimes p_\ell \in (\mathbb{R}^k)^{\otimes \ell}$ we let

$$\langle p_1 \otimes \cdots \otimes p_\ell, p_1 \otimes \cdots \otimes p_\ell \rangle = \sum_{\text{pairings } (i_q,j_p)} \prod \langle p_{i_q}, p_{j_p} \rangle$$

and finally we identify

$$v_1 = (\lambda_1,\ldots,\lambda_k) \oplus (\lambda_{1,2}, \lambda_{1,3}\ldots) \oplus \cdots \in \oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}.$$

Then we deduce that $\langle v_1, v_1 \rangle_{\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}} \geqslant 0$ since it is a positive definite scalar product on $\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}$. We conclude that $\mathbb{E}[Q(X)\overline{Q(\mathbb{R}(X))}] \geqslant 0$.

Now we approximate $\exp(i\alpha X_{\xi_j})$ by polynomials and then we can extend the positivity to convex linear combinations of complex exponentials on $\mathbb{R}^k$. But these are dense in $C_b^0(\mathbb{R}^k, \mathbb{C})$ and therefore we can extend the reflection positivity to all functions in $C_{\text{cyl}}^0(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$. □

Theorem 3. *A gaussian process X satisfies conditions $(1,2,3,4)$ iff r is continuous, translation invariant and such that eq. (1) holds. In the scalar case this holds iff r is completely monotone and bounded and translation invariant.*

Recall that complete monotonocity is exaclty the condition eq. (1) in the scalar case and this implies that there exists a positive and bounded measure μ on $\mathbb{R}_+$ such that

$$r(t) = \int_0^\infty e^{-|t|s} \mu(\mathrm{d}s).$$

Recall that

$$r(t) = \frac{1}{2\theta} e^{-\theta|t|}$$

is the covariance of the Ornstein–Uhlenbeck process. So the theorem says that the reflection positive Gaussian processes are positive combinations of OU processes.

For example, if μ is a sum of Dirac deltas in $(\theta_k)_k$ then one can obtain a Gaussian process with covariance r taking the sum of independent OU processes with parameter θ_k.

Let us now give a look at reflection positivity for Markovian processes.

Definition 4. *A process $(X_t)_{t\in\mathbb{R}}$ is Markovian if $F \in C^0_{\text{cyl}}(M^{[t,+\infty]}, \mathbb{C})$ then for all $\xi_1, \ldots, \xi_k \leqslant t$*

$$\mathbb{E}[F(X)|X_t, X_{\xi_1}, \ldots, X_{\xi_k}] = \mathbb{E}[F(X)|X_t]$$

almost surely.

Definition 5. *The process X is said to be symmetric with respect to time reflections if $\mathbb{R}(X)$ has the same law as X.*

Lemma 6. *If X is Markovian then $F \in C_{\text{cyl}}(M^{[t,+\infty)}, \mathbb{C})$ and $G \in C_{\text{cyl}}(M^{(-\infty,t]}, \mathbb{C})$ then $F(X)$ and $G(X)$ are conditionally independent given X_t.*

Proof. Assuming that $G(X) = \tilde{G}(X_{\xi_1}, \ldots, X_{\xi_k})$ with $\xi_1, \ldots, \xi_k \leqslant t$ we have

$$\mathbb{E}[e^{i\alpha F(X)} e^{i\beta G(X)}|X_t] = \mathbb{E}[\mathbb{E}[e^{i\alpha F(X)}|X_t, X_{\xi_1}, \ldots, X_{\xi_k}] e^{i\beta G(X)}|X_t]$$

$$= \mathbb{E}[\mathbb{E}[e^{i\alpha F(X)}|X_t] e^{i\beta G(X)}|X_t] = \mathbb{E}[e^{i\alpha F(X)}|X_t] \mathbb{E}[e^{i\beta G(X)}|X_t]$$

so this proves conditional independence. □

Theorem 7. *Let X be a Markovian process symmetric with respect to time reflections, then it is reflection positive.*

Proof. Take $F \in C_{\text{cyl}}(M^{\mathbb{R}_+}, \mathbb{C})$ then by the above lemma

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] = \mathbb{E}[\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}|X_0]] = \mathbb{E}[\mathbb{E}[\overline{F(\mathbb{R}(X))}|X_0] \mathbb{E}[F(X)X_0]]$$

$$= \mathbb{E}[\overline{\mathbb{E}[F(\mathbb{R}(X))|X_0]} \mathbb{E}[F(X)X_0]]$$

$$= \mathbb{E}[\overline{\mathbb{E}[F(\mathbb{R}(X))|\mathbb{R}X_0]} \, \mathbb{E}[F(X)X_0]] = \mathbb{E}[|\mathbb{E}[F(X)X_0]|^2] \geqslant 0,$$

where we used that the law is invariant under time reflection. □

The converse implication of the above lemma is also true. Note that the OU process has exactly this property and therefore it means that the OU process is Markovian and since it is symmetric wrt. time reflections we have another proof that that OU process is reflection positive.

If we want to prove the other properties required for the reconstruction theorem we need that X is continuous in distribution and that it is invariant (in law) under translations. These properties can be obtained analysing the transition kernel of the Markov process.

Let us remark that X_t as an M-valued random variable has a law $\nu = \mathrm{Law}(X_t)$ which is independent of $t \in \mathbb{R}$. Then we can build $\mathscr{H} = L^2(\nu)$, with $h_0 = 1$ and $K(t) f \in L^2(\nu)$ is given explictly by

$$\mathbb{E}[f(X_t)|X_0] = (K(t)f)(X_0).$$

The proof is essentially the same we gave for the OU process. The key observation is that if $F \in C_{\mathrm{cyl}}(M^{\mathbb{R}_+}; \mathbb{C})$ we have that

$$\mathbb{E}[(F(X) - \mathbb{E}[F(X)|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F(\mathbb{R}(X))|X_0])}] = 0$$

by Markov property and symmetry under reflections. This allows to identify $\mathscr{H} = L^2(\nu)$ and Q_0 is given by multiplication : $Q_0(a) f = a(x) f(x)$.

Consider a Gaussian process with $r(0) = \mathbb{I}$, then X is Markovian iff $r(t+s) = r(t) r(s)$ (as matrices) for t, $s \geqslant 0$. More generally $r(t,s) = r(t,u) r(u,s)$ for all $s \leqslant u \leqslant t$. So in particular, in the scalar case the process is reflection positive iff it is an OU process.

To construct reflection positive processes which are Markovian but not Gaussian we can take the solution $(X_t)_t$ of a stochastic differential equation of the form

$$\mathrm{d}X_t = \frac{\nabla \rho(X_t)}{2\rho(X_t)} \mathrm{d}t + \mathrm{d}W_t,$$

where $\rho \in C^2(\mathbb{R}^m, \mathbb{R}_{>0})$ and $\int \rho(x) \mathrm{d}x = 1$. And take $\mathrm{Law}(X_{\xi_1}, \ldots, X_{\xi_k})$ to be given by the solution of the SDE starting at X_{ξ_1} with law $\rho \mathrm{d}x$. One can check that this is a consistent assignment of finite dimensional distributions giving a continuous, stationary (i.e. invariant in law under translation), Markov process which is moreover invariant under time reflection. Therefore it defines a reflection positive process to which the reconstruction theorem can be applied. In the case where ρ is Gaussian, then X is the OU process. However if ρ is not Gaussian this procedure gives a large class of reflection positive processes and therefore a large class of quantum dynamics where the Hamiltonian operator H has the form

$$H = -\Delta + V(x)$$

for some function V.

The course ends here.