Lecture 3 – Tue April 28th 2020 – 12:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

C^*-algebras

$a, b, c,$ are arbitrary elements of $\mathscr{A}$ and $\lambda, \mu \in \mathbb{C}$ with $\bar{\lambda}$ the complex conjugate of λ.

Definition 1. *A (unital, always here) C^*-algebra $\mathscr{A}$ is an associative algebra over $\mathbb{C}$, on top which there are a norm $\|*\|$ which makes $\mathscr{A}$ a Banach space and such that $\|ab\| \leqslant \|a\| \|b\|$. There is an antilinear involution $*: \mathscr{A} \to \mathscr{A}$ and such that $(a^*)^* = a$, $(ab)^* = b^* a^*$ $(\lambda a)^* = \bar{\lambda} a^*$. Moreover they satisfy the C^*-condition*

$$\|a^* a\| = \|a\|^2, \qquad a \in \mathscr{A}.$$

We call 1 the unit of $\mathscr{A}$. Note that

$$\|a\|^2 = \|a^* a\| \leqslant \|a^*\| \|a\| \Rightarrow \|a\| \leqslant \|a^*\| \leqslant \|(a^*)^*\| = \|a\|$$

so the involution is isometric and moreove is easy to see that $1^* = 1$ and that $\|1\| = 1$.

Example 2.

- The algebra of complex functions $C(X)$ on a compact Hausdorff space X with sup norm and conjugation is a C^*-algebra.

- The algebra of all bounded operators $\mathscr{L}(\mathscr{H})$ on the Hilbert space $\mathscr{H}$ with the operator norm and the adjoint operation is a C^*-algebra. Let's check this out: take $A \in \mathscr{L}(\mathscr{H})$, then

$$\|A^* A\| = \sup_{\|\varphi\| = \|\psi\| = 1} \langle \psi, A^* A \varphi \rangle = \sup_{\|\varphi\| = \|\psi\| = 1} \langle A\psi, A\varphi \rangle \leqslant \sup_{\|\varphi\| = \|\psi\| = 1} \|A\psi\| \|A\varphi\| \leqslant \|A\|^2$$

  and on the other hand

$$\|A\|^2 = \sup_{\|\varphi\| = 1} \|A\varphi\|^2 = \sup_{\|\varphi\| = 1} \langle A\varphi, A\varphi \rangle = \sup_{\|\varphi\| = 1} \langle \varphi, A^* A\varphi \rangle \leqslant \sup_{\|\varphi\| = \|\psi\| = 1} \langle \psi, A^* A\varphi \rangle = \|A^* A\|$$

  so the C^*-condition holds. Sub-$*$-algebras of $\mathscr{L}(\mathscr{H})$ (i.e. closed by algebraic operations, conjugations and norm convergence) are called *concrete* C^*-algebras (e.g. compact operators).

- The subalgebra $C^*(a) \subseteq \mathscr{A}$ generated by $a \in \mathscr{A}$ and the unity is a C^*-algebra with the restiction of the norm and the involutions of $\mathscr{A}$. The Banach algebra generated by a set of elements $a_1, \ldots, a_n$ is just the closure of all the polynomials in $a_1, \ldots, a_n$ and in their adjoints.

We call a self-adjoint iff $a=a^*$, a is *normal* if $aa^*=a^*a$. Any a can be decomposed into $a=b+ic$ with b,c self-adjoint. If a is normal then $C^*(a)$ is Abelian (i.e. commutative).

Keep in mind that, for us, the observables of a physical system will be self-adjoints elements of an (abstract) C^* algebra.

Example 3. Take $L^1(\mathbb{R})$ or $L^1(\mathbb{R}_{\geqslant 0})$ with the convolution product and their natural norm. Then they are Banach algebras ($\|ab\| \leqslant \|a\| \, \|b\|$). (I think they are not C^*-algebras for the complex conjugation). For the convolution product on $L^1(\mathbb{R}_{\geqslant 0})$ take

$$(f*g)(t)=\int_0^t f(t-s)g(s)\mathrm{d}s, \qquad t\geqslant 0.$$

In the following we will work with Banach algebras (denoted $\mathscr{B}$) and I will tell explicitly when the algebra is supposed to satisfy the C^*-condition.

The spectrum $\sigma(a)\subseteq\mathbb{C}$ of an elements of a Banach algebra $a\in\mathscr{B}$ as the set of all $\lambda\in\mathbb{C}$ for which $\lambda-a=\lambda 1_{\mathscr{B}}-a$ is not invertible in $\mathscr{B}$. The complement of the spectrum is called the resolvent set and $R_a(\lambda)=(\lambda-a)^{-1}$ is defined on $\sigma(a)^c$ and is called resolvent (function) for a.

Theorem 4. *For any $a\in\mathscr{B}$ the spectrum $\sigma(a)$ is a non-empty compact set and the resolvent function is analytic in $\sigma(a)^c$.*

Proof. For λ large we can define

$$(\lambda-a)^{-1}=\frac{1}{\lambda}(1-a/\lambda)^{-1}=\frac{1}{\lambda}\sum_{n\geqslant 0}\left(\frac{a}{\lambda}\right)^n \tag{1}$$

as a convergent series in $\mathscr{B}$ as soon as $\|a\|<\lambda$. It defines an analytic function at infinity and it goes to zero as $|\lambda|\to\infty$. This tells us that $\sigma(a)$ is contained in any ball of radius $>\|a\|$. Moreover if $\mu\notin\sigma(a)$ then $R_a(\mu)=(\mu-a)^{-1}$ exists and we can write

$$R_a(\lambda)=\sum_{n\geqslant 0}(-1)^n(\mu-a)^{-n-1}(\lambda-\mu)^n$$

and have this series converge in a neighb. of μ. So $\sigma(a)^c$ is open therefore $\sigma(a)$ is compact. If $\sigma(a)$ is empty then $R_a(\lambda)$ would be an analytic function on all $\mathbb{C}$ going to zero at infinity, which implies that $R_a(\lambda)=0$ for all λ. $1=(\lambda-a)R_a(\lambda)=0$. □

Proposition 5. *(Spectral radius formula) For any $a\in\mathscr{B}$*

$$\varrho(a):=\sup_{\lambda\in\sigma(a)}|\lambda|=\lim_{n\to\infty}\|a^n\|^{1/n}\leqslant\|a\|.$$

Moreover if $\mathscr{B}$ is a C^* algebra and a is normal then there is equality on the r.h.s.

Remark 6. This shows that C^* are quite rigid, in the sense that the algebraic data defines the norm. The quantity $\varrho(a)$ is called the spectral radius of a.

Proof. First step is to prove that the limit $r = \lim_{n\to\infty} \|a^n\|^{1/n}$ exists, then by the convergence of the resolvent series (1) one prove that $\varrho(a) = r$. For a C^* algebra we have now that if a is normal we have

$$\|a^2\|^2 \underset{C^*}{=\!=} \|a^*a^*aa\| \underset{\text{normality}}{=\!=\!=\!=} \|aa^*a^*a\| = \|(a^*a)^*a^*a\| \underset{C^*}{=\!=} \|a^*a\|^2 \underset{C^*}{=\!=} \|a\|^4.$$

So $\|a^{2^k}\| = \|a\|^{2^k}$ and therefore $\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|$. □

Remark 7. We will see later that $\varrho(a) = \|a\|$ for any $a \in \mathscr{A}$.

Let $\mathscr{B}^*$ be the space of continuous linear functionals $\varphi: \mathscr{B} \to \mathbb{C}$ on $\mathscr{B}$. We can consider $\mathscr{B}^*$ as a Banach space with norm $\|\varphi\| = \sup_{\|a\|=1} |\varphi(a)|$. The weak-$*$ topology on $\mathscr{B}^*$ is the coarsest topology on $\mathscr{B}^*$ which makes all the linear functionals $a: \varphi \in \mathscr{B}^* \mapsto \varphi(a)$ continuous for all $a \in \mathscr{B}$. By Banach-Alaoglu theorem the unit ball of $\mathscr{B}^*$ is weakly-$*$ compact.

A linear functional φ is multiplicative if

$$\varphi(ab) = \varphi(a)\varphi(b),$$

(also called a character). Any multiplicative functional is bounded (and therefore continuous). Indeed let $a \in \mathscr{B}$ such that $\|a\| < 1$ but $\varphi(a) = 1$. Then there always exists $b = 1 + ab$ (think why, fixpoint). The $\varphi(b) = \varphi(1) + \varphi(a)\varphi(b)$ which implies since $\varphi(a) = 1$ that $\varphi(1) = 0$. But now $\varphi(c) = \varphi(c1) = \varphi(c)\varphi(1) = 0$. So $\varphi = 0$. Therefore we know that $\varphi(a) \leqslant \|a\|$ and that $\|\varphi\| \leqslant 1$ and since $\varphi(1) = 1$ we have $\varphi(1) = \|\varphi\|$.

So multiplicative functionals sits on the unit ball of $\mathscr{B}^*$. Let $\Sigma(\mathscr{B}) \subseteq \mathscr{B}^*$ to the space of all linear multiplicative functionals on $\mathscr{B}$. The set $\Sigma(\mathscr{B})$ is not a vector space.

Proposition 8. *The space $\Sigma(\mathscr{B})$ is a compact Hausdorff space when endowed with the weak-$*$ topology of $\mathscr{B}^*$. It is called the Gelfand spectrum of $\mathscr{B}$.*

For any $a \in \mathscr{B}$ we can define the function $\hat{a}: \Sigma(\mathscr{B}) \to \mathbb{C}$ (called the Gelfand transform of a) by $\hat{a}(\varphi) = \varphi(a)$. Is a continuous function due to the weak-$*$ topology.

Theorem 9. *The Gelfand transform is a contractive algebra homeomorphism from $\mathscr{B}$ to $C(\Sigma(\mathscr{B}))$. If $\mathscr{B}$ is an Abelian C^*-algebra then the Gelfand transform is an isomorphism.*
