Lecture 4 – Wed April 29th 2020 – 8:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

C^*-algebras (continued)

$\mathcal{A}$ is a C^* algebra: C^*-condition: $\|a^*a\| = \|a\|^2$. Self-adjoints $a = a^*$. Normal operators $a^*a = aa^*$. Any a can be decomposed $a = b + ic$ with b, c self-adjoint but not necessarily commuting unless the element is normal.

Spectrum $\lambda \in \sigma(a) \subseteq \mathbb{C} \Leftrightarrow (\lambda - a)$ is not invertible in $\mathcal{A}$ $\Leftrightarrow (\lambda - a)$ in any subalgebra $\mathcal{A}' \subseteq \mathcal{A}$ containing a. In particular we can use $C^*(a)$ to compute the spectrum of a.

The resolvent $R_\lambda(a) = (\lambda - a)^{-1} \in C^*(a)$ is analytic in $\sigma(a)^c$.

$\mathcal{B}^*$: continuous linear functionals on a the Banach algebra $\mathcal{B}$. (norm $\|\varphi\| = \sup_{a \in \mathcal{B}} |\varphi(a)|$, also weak-$*$ topology)

Multiplicative functional $\varphi: \mathcal{B} \to \mathbb{C}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$. It is continuous and of norm 1: $|\varphi(a)| \leqslant \varphi(1) = 1$ so $\|\varphi\| = 1$. Space of linear multiplicative functionals $\Sigma(\mathcal{B})$ (Gelfand spectrum).

Commutator $[a, b] := ab - ba$. Note that $\varphi([a, b]) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$.

$\Sigma(\mathcal{B})$ is a compact Hausdorff space. And the Gelfand transform of a, that is $\hat{a}(\varphi) = \varphi(a)$ is a continuous function on $\Sigma(\mathcal{B})$.

$$a \in \mathcal{B} \mapsto \hat{a} \in C(\Sigma(\mathcal{B})).$$

Moreover $\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \hat{a}(\varphi)\hat{b}(\varphi)$ so the Gelfand transform is an algebra homomorphism and taking the sup norm on $C(\Sigma(\mathcal{B}))$ we have

$$\|\hat{a}\|_{C(\Sigma(\mathcal{B}))} = \sup_{\varphi \in \Sigma(\mathcal{B})} |\hat{a}(\varphi)| = \sup_{\varphi \in \Sigma(\mathcal{B})} |\varphi(a)| \leqslant \sup_{\varphi \in \Sigma(\mathcal{B})} \|\varphi\| \|a\| = \|a\|.$$

The map $a \mapsto \hat{a}$ is contractive.

Remark 1. If $\mathcal{B} = (L^1(\mathbb{R}), *)$ the Gelfand transform is Fourier transform. While for $(L^1(\mathbb{R}_+), *)$ the transform is the Laplace transform.

We need to ensure that $C(\Sigma(\mathcal{B}))$ looks quite like $\mathcal{B}$. In order for the transformation to be useful we will require that $\mathcal{B}$ is commutative (Abelian).

Theorem 2. *For a commutative Banach algebra $\mathcal{B}$ multiplicative linear functionals corresponds to maximal proper ideals of $\mathcal{B}$ (one to one).*

$\varphi(a)=0$ then $\varphi(ab)=0$ so $\ker(\varphi)$ is an ideal of $\mathscr{B}$, i.e. $a\mathscr{B}\subseteq\ker(\varphi)$ for any $a\in\ker(\varphi)$. In general from an ideal one cannot costruct a multiplicative functionals. But if the ideal $\mathscr{J}$ is maximal and proper (i.e.$1\notin\mathscr{J}$) then one can show that it closed and that the quotient $\mathscr{B}\backslash\mathscr{J}$ (that is $[a]\in\mathscr{B}\backslash\mathscr{J}$ is $[a]=a+\mathscr{J}$) is still a Banach algebra and that in the quotient all the elements (apart zero) are invertible (due to the maximality of $\mathscr{J}$) (think about it) and due to this fact one can prove that $\mathscr{B}\backslash\mathscr{J}=\mathbb{C}$. (the spectrum of an element cannot be empty but all the elments apart from zero are invertible so one has that $\lambda-a=0$). So given a maximal proper ideal one can construct a multiplicative functional φ by letting $\varphi=0$ on $\mathscr{J}$ and $\varphi(1)=1$. This defines φ on all $\mathscr{B}=\mathscr{J}+\mathbb{C}$.

Example 3. Take $\mathscr{B}=C(X)$. Then multiplicative functionals are just δ_x for any $x\in X$ and maximal proper ideals are $\mathscr{J}_x=\{f\in C(X): f(x)=0\}$.

Now take a Abelian $\mathscr{B}$ and let a be an invertible element and $\varphi\in\Sigma(\mathscr{B})$, therefore

$$1=\varphi(aa^{-1})=\varphi(a)\varphi(a^{-1})=\hat{a}(\varphi)\widehat{a^{-1}}(\varphi)$$

therefore $\hat{a}$ is invertible in $C(\Sigma(\mathscr{B}))$ and the other way around. This implies that

$$\sigma_{\mathscr{B}}(a)=\sigma_{C(\Sigma(\mathscr{B}))}(\hat{a})=\{\hat{a}(\varphi):\varphi\in\Sigma(\mathscr{B})\}=\{\varphi(a):\varphi\in\Sigma(\mathscr{B})\}$$

[$\lambda\in\sigma_{C(\Sigma(\mathscr{B}))}(\hat{a})$ if $\lambda-a$ is not invertible, one can prove that $\mathscr{J}=(\lambda-a)\mathscr{B}$ is a maximal proper ideal and if φ is the associated multiplicative functional that is zero on $\mathscr{J}$ so $0=\varphi(\lambda-a)$ and $\lambda=\varphi(a)$]

Therefore

$$\|a\|\geqslant\varrho(a)=\varrho(\hat{a})=\sup\{|\hat{a}(\varphi)|:\varphi\in\Sigma(\mathscr{B})\}=\|\hat{a}\|_{C(\Sigma(\mathscr{B}))}.$$

Theorem 4. *(Gelfand–Naimark) If $\mathscr{A}$ is a commutative C^*-algebra then $\mathscr{A}\approx C(\Sigma(\mathscr{A}))$ (isometrically isomorphic).*

We have to check that the involution transform correctly, namely $\widehat{a^*}=\bar{\hat{a}}$ i.e.

$$\varphi(a^*)=\overline{\varphi(a)}$$

for any $\varphi\in\Sigma(\mathscr{A})$. Assume a is self-adjoint, then we can form (by convergent series in the Banach algebra)

$$U(t)=\exp(iat)=\sum_{n\geqslant 0}\frac{(it)^n}{n!}a^n$$

and check that it is unitary in $\mathscr{A}$, i.e. $U(-t)=U(t)^*$ and $U(-t)U(t)=\exp(-iat)\exp(iat)=1$.

$$\varphi(U(t))=\sum_{n\geqslant 0}\frac{(it)^n}{n!}\varphi(a^n)=\sum_{n\geqslant 0}\frac{(it)^n}{n!}\varphi(a)^n=\exp(it\varphi(a))$$

but now

$$|\varphi(U(t))|\leqslant\|\varphi\|\,\|U(t)\|=\|U(t)\|=1$$

since $\|U(t)\|^2 = \|U(t)^* U(t)\| = \|1\| = 1$. Therefore $|\exp(it\varphi(a))| \leqslant 1$ for all $t \in \mathbb{R}$ and this implies that $\varphi(a) \in \mathbb{R}$. For a general $a = b + ic$ then $\varphi(a) = \varphi(b) + i\varphi(c)$ and

$$\varphi(a^*) = \varphi(b - ic) = \varphi(b) - i\varphi(c) = \overline{\varphi(a)}.$$

Remember that for C^*-algebras we have that if a is normal then $\|a\|_{\mathscr{A}} = \varrho_{\mathscr{A}}(a)$ therefore we have

$$\|a\|_{\mathscr{A}} = \|\hat{a}\|_{C(\Sigma(\mathscr{B}))}.$$

Now use againg the C^*condition to get for any $a \in \mathscr{A}$ (observe that $a^* a$ is self-adjoint)

$$\|a\|^2 \underset{C^*}{=\!=} \|a^* a\| = \|\widehat{a^* a}\| = \|\widehat{a^*}\, \hat{a}\|_\infty = \|\bar{\hat{a}}\, \hat{a}\|_{C(\Sigma(\mathscr{A}))} \underset{C^*}{=\!=} \|\hat{a}\|^2_{C(\Sigma(\mathscr{A}))}$$

so we conclude that the transform is an isomorphism. It is one to one since if $\varphi(a) = \varphi(b)$ for all φ then $\varphi(a - b) = 0$ for all φ, this implies that $\|a - b\| = 0$.

This concludes the proof of the theorem.

Exercise 1. Take the commutative C^* algebra $\mathscr{A}$ of diagonal $n \times n$ matrices. Prove that it is a C^*-algebra with the structure inherited from the space of all matrices, i.e. norm is the operator norm, involution is the adjoint, product is product of matrices. Try to work out the space $\Sigma(\mathscr{A})$.

By GN theorem we now know that for any $a \in \mathscr{A}$ if $\mathscr{A}$ is commutative we have

$$\varrho(a) = \varrho(\hat{a}) = \|\hat{a}\| = \|a\|$$

which coincide with what we knew before for normal elements.

The GN theorem give us a continuous functional calculus on normal elements of a general C^* algebra. Indeed observe that for any normal $a \in \mathscr{A}$, the C^*-algebra $C^*(a)$ generated by a is commutative then there is an isomorphism

$$C^*(a) \approx C(\Sigma(C^*(a))) \approx C(\sigma(a))$$

Since is not difficult to show that $\Sigma(C^*(a)) = \sigma(a) \subseteq \mathbb{C}$. So if you give me a continuous function $f: \sigma(a) \to \mathbb{C}$ then this defines an element $f(a) \in C^*(a)$ by

$$\widehat{f(a)} = f.$$

Moreover $f(g(a)) = (f \circ g)(a)$, etc.. $f(a)$ is selfadjoint if f is a real function, etc…

Next week we will discuss positivity in C^* (i.e. when $a \geqslant 0$) and positive linear functionals.

Need non-commutativity (Heisenberg indetermination principle).

Representation theory for non-commutative C^*-algebras on Hilbert spaces.