Lecture 5 – Tue May 5th 2020 – 14:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

[P. Meier, *Quantum probability for probabilists*. Springer. Very nice appendix on C^*-algebra]

C^*-algebras (continued)

Positivity, state (positive linear functionals), GNS theorem, Hilbert space setting for QM. Heisenberg indetermination principle.

$$\sigma(a) = \{\lambda \in \mathbb{C} : (\lambda - a) \text{is not invertible}\} \subseteq \mathcal{A}$$

$$\varrho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \lim_n \|a^n\|^{1/n}.$$

If a is normal then

$$\varrho(a) = \|a\|.$$

Spectral mapping principle: Let $a \in \mathcal{A}$. If $f: \mathbb{C} \to \mathbb{C}$ is holomorphic in the neighborhood of $\sigma(a)$ then $f(a)$ is well defined (by series expansion) and $\sigma(f(a)) = f(\sigma(a))$. In particular this holds for polynomials.

Definition 1. *A self-adjoint element $a \in \mathcal{A}$ is positive iff $\sigma(a) \subseteq \mathbb{R}_{\geqslant 0}$. We write $a \geqslant 0$ or $a \in \mathcal{A}_+$.*

Some properties

- Any self-adjoint positive a can be written as $a = b^2$ where $b = a^{1/2}$ (by functional calculus).
- If $c^2 = a = b^2$ with s.a. $b, c \in \mathcal{A}_+$ then $b = c = a^{1/2}$. There is only one positive square root.
- One can decompose any s.a. a into the difference of two positive elements $a = a_+ - a_-$ (by functional calculus).

What is the structure of $\mathcal{A}_+$?

- If a is self adjoint, $\|a\| \leqslant 1$ then $a \geqslant 0$ iff $\|1 - a\| \leqslant 1$. Observe that $\sigma(1 - a) = 1 - \sigma(a)$, $\varrho(1 - a) \leqslant 1$ and $\varrho(a) \leqslant 1$ therefore $\sigma(a) \subseteq [0, 1]$.
- If $a \geqslant 0$ then $\lambda a \geqslant 0$ for all $\lambda \geqslant 0$ since $\sigma(\lambda a) = \lambda \sigma(a)$. So $\lambda \mathcal{A}_+ \subseteq \mathcal{A}_+$ and $\mathcal{A}_+$ is a cone.
- Convex combinations of positive elements are positive: $a = \lambda b + (1 - \lambda) c$, with $b, c \geqslant 0$ and $\lambda \in [0, 1]$ then assume that $\|b\|, \|c\| \leqslant 1$ so $\|a\| \leqslant \lambda \|b\| + (1 - \lambda)\|c\| \leqslant 1$

$$\|1 - a\| \leqslant \lambda \|1 - b\| + (1 - \lambda)\|1 - c\| \leqslant \lambda + (1 - \lambda) \leqslant 1$$

so $a \geqslant 0$. Therefore $\mathscr{A}_+$ is convex and by similar reasoning one can show that $\mathscr{A}_+$ is closed.

Observe that if $\mathscr{A}_+ = \mathscr{L}(H)$ then the element A^*A is positive in the operator sense $\langle x, A^*Ax\rangle \geqslant 0$ for $x \in H$. From this follows also that it is positive in the sense of C^* algebras.

For long time is was conjectured that this was true for any C^* algebra (namely that elements of the form a^*a are positive). But was not proven easily.

Theorem 2. *For $a \in \mathscr{A}$ we have $a^*a \geqslant 0$ (and all the positive elements have that form).*

We write

$$a \geqslant b \Leftrightarrow a - b \geqslant 0.$$

Beware of "trivial" inequalities because they could not be true in general C^* algebras.

For example if $0 \leqslant a \leqslant b$ then is not true in general that $a^2 \leqslant b^2$. If you define $|a| = (a^*a)^{1/2}$ then is not true in general that $|a+b| \leqslant |a| + |b|$.

True inequalities:

- $a \leqslant \|a\|$ (i.e. $\|a\|1 - a \in \mathscr{A}_+$) and $a^2 \leqslant \|a\|a$ (by spectral considerations). $f(x) = \|a\| - x \geqslant 0$ on $\sigma(a)$.
- $a \geqslant 0 \Rightarrow c^*ac \geqslant 0$ for any $c \in \mathscr{A}$. (by Theorem 2)
- $a \geqslant b \Rightarrow f(a) \geqslant f(b)$ when $f(x) = x^\alpha$ for $\alpha \in (0,1]$ and when $f(x) = (\lambda - x)^{-1}$ with $\lambda > 0$ and so in general for functions of the form

  $$f(x) = \int_{\mathbb{R}_+} (\lambda - x)^{-1} \mu(\mathrm{d}\lambda)$$

  for any (positive) measure μ on $\mathbb{R}_+$.

States on C^* algebras

Definition 3. *A linear functional $\omega: \mathscr{A} \to \mathbb{C}$ is positive if $\omega(a) \geqslant 0$ for all $a \geqslant 0$.*

If ω is positive the the Cauchy–Schwarz inequality holds

$$|\omega(a^*b)| \leqslant \omega(a^*a)\omega(b^*b), \qquad a, b \in \mathscr{A}.$$

(exercise, like the standard proof). Note that

$$\langle a, b\rangle_\omega = \omega(a^*b)$$

is a positive-definite Hermitian form on $\mathscr{A}$ (antilinear on the left and linear on the right) for any positive linear functional.

Proposition 4. *Any bounded linear functional* $\omega\colon \mathcal{A}\to\mathbb{C}$ *with* $\|\omega\| = \omega(1) = 1$ *satisfy*

$$\omega(a^*) = \overline{\omega(a)}.$$

Proof. It is enough to prove it for s.a. a. Namely that $\omega(a)\in\mathbb{R}$. You observe that

$$|\omega(f(a))| \leqslant \|\omega\| \|f(a)\| = \|\omega\| \sup\{|f(x)|: x\in\sigma(a)\} = \omega(1)\, \|f\|_{C(\sigma(a))}$$

This tells us that $f\mapsto \ell(f) = \omega(f(a))$ is a continous linear functional on continuous functions with the uniform norm on the compact set $\sigma(a)\subseteq\mathbb{C}$. So $\ell(f) = \int_{\sigma(a)} f(x)\,\mu(\mathrm{d}x)$ for some Randon measure μ and any such measure has to be positive so $\omega(a) = \int_{\sigma(a)} x\mu(\mathrm{d}x)\in\mathbb{R}$. □

Proposition 5. *A linear functional is positive iff* $\|\omega\| = \omega(1)$.

Proof. If ω is positive use that $\|a\| - a \geqslant 0$ and that $\|a\| + a \geqslant 0$ to deduce that $0 \leqslant \omega(\|a\| - a)$ i.e. $\omega(a) \leqslant \|a\|\omega(1)$ and similarly $\omega(a) \geqslant -\|a\|\omega(1)$. Observe also that $\omega(1)\geqslant 0$. so

$$|\omega(a)| \leqslant \|a\|\, \omega(1),$$

so that $\|\omega\| = \omega(1)$. On the other hand if ω is bounded and $\|\omega\| = \omega(1)$. I can assume that $\omega(1) = 1$, the for any $a\geqslant 0$, $\|a\|\leqslant 1$ we have $\omega(1-a) = 1-\omega(a)$ so

$$|1-\omega(a)| = |\omega(1-a)| \leqslant \|1-a\| \leqslant 1$$

moreover $\overline{\omega(a)} = \omega(a^*) = \omega(a)\in\mathbb{R}$ and we conclude that $\omega(a)\in[0,1]$. □

Remark: multiplicative functionals are positive.

We in general call a state a normalized positive linear functional. Is important to observe that there are enough states to separate the elements of $\mathcal{A}$ and that an element is positive iff gives positive value on every state.

Proposition 6. *Positive linear functionals separate* $\mathcal{A}$ *and if* $\omega(a)\geqslant 0$ *for all positive* ω *then* $a\in\mathcal{A}_+$

Proof. (sketch) First part. Assume that $\omega(a) = 0$ for any state ω. We can assume a to be s.a. (by taking real and imaginary part) then by Gelfand spectral theory we know that $\|a\| = \|\hat{a}\| = \{\varphi(a)\colon \varphi\in\Sigma(\mathcal{A})\}$ but then $\varphi(a) = 0$ because φ is a state and therefore $\|a\| = 0$. Second part. $\omega(a)\geqslant 0$ implies that $\omega(a) = \overline{\omega(a^*)} = \omega(a^*)$ this implies that $\omega(a - a^*) = 0$ by separation we have that $a = a^*$ and by Gelfand transform we deduce $\sigma(a) = \sigma(\hat{a}) \subseteq \mathbb{R}_+$. □

Therefore the notion of positivity based on observational requirements (i.e $\omega(a)\geqslant 0$ for any state) coincide with the "abstract" notion of positivity given by $\sigma(a)\subseteq\mathbb{R}_+$. The interest and motivation to set up things in a C^* algebra comes from this nice relations between states and observables.

The set of states ω is a convex closed subset of all the linear functinals, in particular since $\|\omega\| = 1$ then it is contained in the closed unit ball and therefore is a compact subset in the weak-* topology. Any state can be decomposed as convex combination of pure states where a pure state is a a state which does not allow such a decomposition.

Example: if $\mathscr{A} = \mathscr{L}(H)$ then any $x \in H$ such that $\|x\|_H = 1$ define a pure state $\omega(A) = \langle x, Ax \rangle$, this kind of states are called state vectors.

Tomorrow we discuss the GNS theorem and we motivate the use of Hilbert space in the study of QM.