Lecture 6 – Wed May 6th 2020 – 8:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

[P. Meier, *Quantum probability for probabilists*. Springer. Very nice appendix on C^*-algebra]

C^*-algebras – GNS Theorem (continued)

Positivity, state (positive linear functionals), GNS theorem, Hilbert space setting for QM.

So far we conceptualized the basic structure of a physical system and the related observation and measurement theory (algebra of observables and the convex set of state of physical system). This applies both to classical and quantum (i.e. non-classical) systems. We also argued that a classical system is given by an algebra of observables given by continuous functions on a “state space”. For the moment anything escaping this point of view will be *quantum* therefore we need to take a non-commutative algebra (by the Gelfand-Naimark theorem).

How do we do computations in a non-commutative C^* algebra. We need (concrete) representations of non-commutative C^*-algebras in order to use the theory to make prediction and compare to experiments.

That's goal for today.

But for the moment let us go back to some loose end from yesterday.

Proposition 1. *Let ω is a continuous linear functional such that $\|\omega\| = \omega(1) = 1$ then $\omega(a^*) = \overline{\omega(a)}$.*

Proof. We can assume that a is s.a. since then is easy to conclude. Assume that $\omega(a) = f + ig$ with $f, g \in \mathbb{R}$ I need to prove that $g = 0$. Take $a + ic$ with $c \in \mathbb{R}$ and observe that $(a + ic)^*(a + ic) = a^2 + c^2$ then $\omega(a + ic) = f + i(g + c)$ so

$$f^2 + (g + c)^2 = |\omega(a + ic)|^2 \leqslant \|a + ic\|^2 \underset{C^*}{=} \|(a + ic)^*(a + ic)\| = \|a^2 + c^2\| \leqslant \|a^2\| + c^2 \leqslant \|a\|^2 + c^2.$$

Now c is arbitrary so we get that $g^2 + 2gc \leqslant \|a\|^2$ which is impossible unless $g = 0$. □

The Gelfand–Naimark–Segal theorem allows to construct representations of C^* algebras on an Hilbert space starting from any state ω (i.e. normalized positive linear functional).

Namely we want to construct a map $\varphi: \mathscr{A} \to \mathscr{L}(H)$ for some complex Hilbert space H such that φ is linear, $\varphi(1) = 1$, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ where on the r.h.s. the involution is understood as the adjoint in the Hilbert space. This is also called a $*$-homomorphism.

Remark 2. Any multiplicative functional $\varphi: \mathscr{A} \to \mathbb{C}$ give a one-dimensional representation on the Hilbert space $H = \mathbb{C}$.

Let us observe that any such representation is necessarily a contraction. Indeed note that if $\lambda - a$ is invertible in $\mathcal{A}$ then exists $c \in \mathcal{A}$ s.t. $c(\lambda - a) = 1$ that implies $\varphi(c)(\lambda - \varphi(a)) = 1$ so $\lambda - \varphi(a)$ is also invertible, that is $\sigma_{\mathcal{L}(H)}(\varphi(a)) \subseteq \sigma_{\mathcal{A}}(a)$. So for C^*-algebras

$$\|\varphi(a)\|_{\mathcal{L}(H)} \underset{C^*}{=} \varrho_{\mathcal{L}(H)}(\varphi(a)) \leqslant \varrho_{\mathcal{A}}(a) \underset{C^*}{=} \|a\|.$$

If φ is an isomorphism (on his image), i.e. $\ker(\varphi) = \{0\}$ on has that φ is an isometry since φ^{-1} is another representation and $\|a\| = \|\varphi^{-1}(\varphi(a))\| \leqslant \|\varphi(a)\| \leqslant \|a\|$.

Let ω be a state and consider the complex pre-Hilbert space $(\mathcal{A}, \langle \cdot, \cdot \rangle_\omega)$ where the Hermitian scalar product is given by

$$\langle a, b \rangle_\omega = \omega(a^* b).$$

This scalar product is positive semi-definite, i.e. $\omega(a^* a) \geqslant 0$. Let $\mathcal{N}_\omega = \{x \in \mathcal{A}: \omega(a^* a) = 0\}$ and let $H_\omega = \overline{\mathcal{A} \backslash \mathcal{N}_\omega}^{\omega}$ the Hilbert space obtained by completion of $\mathcal{A}$ and quotienting of $\mathcal{N}_\omega$. Let $\varphi_\omega(a) b = ab$ for any $\mathcal{A}$ and observe that

$$\langle \varphi_\omega(a) b, \varphi_\omega(a) b \rangle_\omega = \langle ab, ab \rangle_\omega = \omega((ab)^* ab) = \omega(b^* a^* ab) \leqslant \|a\|^2 \omega(b^* b) \leqslant \|a\|^2 \|b\|_{H_\omega}$$

since $a^* a \leqslant \|a^* a\| = \|a\|^2$ and $b^* a^* ab \leqslant b^* \|a\|^2 b = \|a\|^2 b^* b$. So $\varphi_\omega(a) \mathcal{N}_\omega \subseteq \mathcal{N}_\omega$ and $\varphi_\omega(a)$ is a well defined bounded operator on $\mathcal{A} \backslash \mathcal{N}_\omega$ which can then extended by continuity to all H_ω. It is clear that φ_ω is a representation of $\mathcal{A}$ on $\mathcal{L}(H_\omega)$.

This is called the Gelfand-Naimark-Segal representation of $\mathcal{A}$ associated to the state ω. Note that there is a vector $\Omega_\omega = 1_{\mathcal{A}} \in H_\omega$ called the "vaccuum vector" and that we have

$$\omega(a) = \langle \Omega_\omega, \varphi_\omega(a) \Omega_\omega \rangle_\omega.$$

therefore the state ω is represented on H_ω as a vector state.

Moreover the linear space $\{\varphi_\omega(a) \Omega_\omega : a \in \mathcal{A}\}$ is dense in H_ω therefore one says that the vector Ω_ω is cyclic and that a represenation with a cyclic vector is cyclic.

Let K be another Hilbert space supporting another representation $\pi: \mathcal{A} \to \mathcal{L}(K)$ of $\mathcal{A}$ with cyclic vector $\psi \in K$ i.e. $\{\pi(a)\psi : a \in \mathcal{A}\}$ is dense in K and such that $\omega(a) = \langle \psi, \pi(a) \psi \rangle_K$ then there exists a unitary map $U: H_\omega \to K$ such that $\varphi_\omega(a) = U^{-1} \pi(a) U$. (exercise). This tells us that the GNS representation is unique (up to isomorphism).

Theorem 3. *(GNS theorem) For any state ω there exist a cyclic representation φ_ω of $\mathcal{A}$ in Hilbert space H_ω with a vector Ω_ω such that*

$$\omega(a) = \langle \Omega_\omega, \varphi_\omega(a) \Omega_\omega \rangle.$$

Side remarks

- φ_ω acts as left multiplication on $\mathcal{A}$. We can also define $R_a b = ba$ but then on $\mathcal{A}$ this do not give in general a bounded operator (on the Hilbert space H_ω).

- A state is *faithful* is $\omega(a^*a) = 0 \Rightarrow a = 0$.

- A representation φ is faithful if $\ker \varphi = \{0\}$.

- Think about the GNS construction for a commutative algebra. In this case $\mathscr{A} = C(X)$ and ω is a probability measure μ on X so $H_\omega = L^2(X, \mu)$ and if μ is not supported on the full X the representation is not faithful.

There exists a way to construct a faithful representation in Hilbert space. It's the content of the (non-commutative) Gelfand–Naimark theorem.

Take $H = \oplus_{\omega \in \mathscr{S}} H_\omega$ meaning that $x \in H$ is the family $x = (x_\omega)_{\omega \in \mathscr{S}}$ with $x_\omega \in H_\omega$ and scalar product

$$\langle x, y \rangle = \sum_{\omega \in \mathscr{S}} \langle x_\omega, y_\omega \rangle_\omega.$$

Let $\varphi(a)x = (\varphi_\omega(a)x_\omega)_{\omega \in \mathscr{S}}$. We have

$$\|\varphi(a)x\|_H^2 = \sum_{\omega \in \mathscr{S}} \|\varphi_\omega(a)x_\omega\|_{H_\omega}^2 \leqslant \|a\|^2 \sum_{\omega \in \mathscr{S}} \|x_\omega\|_{H_\omega}^2 = \|a\|^2 \|x\|_H^2 \qquad a \in \mathscr{A}, x \in H.$$

So φ is representation of $\mathscr{A}$ in H and if $\varphi(a) = 0$ this means that for any $x \in H$,

$$0 = \|\varphi(a)x\|_H = \sum_{\omega \in \mathscr{S}} \|\varphi_\omega(a)x_\omega\|_{H_\omega}^2$$

so $\|\varphi_\omega(a)x_\omega\|_{H_\omega} = 0$ for any $\omega \in \mathscr{S}$. Take $x_\omega = \Omega_\omega$ so $\omega(a^*a) = 0$ for any $\omega \in \mathscr{S}$ and this implies that $a^*a = 0$ and therefore $0 = \|a^*a\| \underset{C^*}{=} \|a\|^2$ so $a = 0$.

Is enough to use a subset of states which separate points in $\mathscr{A}$. So if $\mathscr{A}$ is separable then H is separable since $\mathscr{S}$ can be taken countable.

Next week, I will discuss matters of irreducibility and pure states. And the Heisenberg indetermination principle and the canonical commutation relations which give rise to a quantum particle.