Lecture 7 – Tue May 11th 2020 – 14:15 via Zoom – M. Gubinelli

[Strocchi, F. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. 2 edition. New Jersey: World Scientific Publishing Company, 2008.]

[P. Meyer, *Quantum probability for probabilists*. Springer. Very nice appendix on C^*-algebra]

C^*-algebras – GNS Theorem, representations (continued)

Indeed is clear that if you have two representations φ, ψ acting on two Hilbert spaces H_1, H_2 you can alway form another representation $\varphi \otimes \psi$ acting on the tensor product Hilbert space $H_1 \otimes H_2$ (revise the definition) as

$$(\varphi \otimes \psi)(a)(v \otimes w) = \varphi(a)v \otimes \psi(a)w \qquad v \in H_1, w \in H_2.$$

Definition 1. *A representation φ is irreducible if the only invariant subspaces for the action of the family of operators $\varphi(\mathscr{A}) = \{\varphi(a): a \in \mathscr{A}\} \subseteq \mathscr{L}(H)$ are $\{0\}, H$.*

For any family $\mathscr{B} \subseteq \mathscr{L}(H)$ we denote by $\mathscr{B}'$ the commutant of $\mathscr{B}$, that is the set

$$\mathscr{B}' = \{C \in \mathscr{L}(H): [C, B] = 0, \forall B \in \mathscr{B}\},$$

where $[C, B] = CB - BC$. Note that $\mathscr{B} \subseteq \mathscr{B}''$ and that $\mathscr{B}' \supseteq \mathbb{C} = \{\lambda \mathbb{1}: \lambda \in \mathbb{C}\}$.

Lemma 2. *The representation φ is irreducible iff $\varphi(\mathscr{A})' = \mathbb{C} = \{\lambda \mathbb{1}: \lambda \in \mathbb{C}\}$.*

Proof. If φ is reducible then let P the orthogonal projection on a non-trivial invariant subspace. Let $v \in PH$ then we have $\varphi(a)v \in PH$ and $\varphi(a)Pv = \varphi(a)v = P\varphi(a)v$. If $v \notin PH$ then $v \in QH$ with $Q = 1 - P$ and then for any w

$$\langle w, \varphi(a)Qv \rangle = \langle \varphi(a)^* w, Qv \rangle = \langle \varphi(a)^*(P + Q)w, Qv \rangle$$

$$= \langle P\varphi(a^*)w, Qv \rangle + \langle \varphi(a)^* Qw, Qv \rangle = \langle Qw, \varphi(a)v \rangle = \langle w, Q\varphi(a)v \rangle$$

so $[\varphi(a), Q] = 0$. Then is clear that $P \in \varphi(\mathscr{A})'$. Reciprocally if $H \in \varphi(\mathscr{A})'$ is a nontrivial self-adjoint element of $\mathscr{L}(H)$, by spectral calculus we can produce a projection $P \in \varphi(\mathscr{A})'$ by setting $P = \chi(H)$ with $\chi: \mathbb{R} \to \mathbb{R}$ some characteristic function of a subset of $\mathbb{R}$, then $P^2 = P$ so P is indeed a projection and the associated subspace is invariant under $\varphi(\mathscr{A})$ since P commute with any $\varphi(a)$. □

Remark 3. Remember that a representation φ is cyclic if there exists a vector $v \in H$ such that $\{\varphi(a)v: a \in \mathscr{A}\}$ is dense in H. Note that any irreducible representation is cyclic for any vector. However not all the cyclic representations are irreducible (see below).

Recall that the set of all states $\mathscr{S}$ (positive normalized linear functionals on $\mathscr{A}$) is a convex set, closed for the weak-$*$ topology. A *pure state* is (by definition) an extremal point in this convex set, i.e. cannot be written as linear combination of other states. By Krein–Milman theorem the set of all states is the closure (in the weak-$*$ topology) of the convex combitations of pure states.

Proposition 4. *The GNS representation* $(H_\omega, \varphi_\omega, \Omega_\omega)$ *for a state* ω *is irreducible iff the state* ω *is pure.*

Proof. Let's assume that φ_ω is reducible, that is there exists a non-trivial orthogonal projection P in $\varphi_\omega(\mathscr{A})'$, then observe that, with $\Omega_\omega \in H_\omega$ the vacuum vector for φ_ω and with $Q = 1 - P$

$$\omega(a) = \langle \Omega_\omega, \varphi_\omega(a)\Omega_\omega\rangle_{H_\omega} = \langle P\Omega_\omega, \varphi_\omega(a)P\Omega_\omega\rangle_{H_\omega} + \langle Q\Omega_\omega, \varphi_\omega(a)Q\Omega_\omega\rangle_{H_\omega},$$

where the cross terms disappear since P commutes with $\varphi_\omega(a)$. Observe that

$$\lambda = \langle P\Omega_\omega, P\Omega_\omega\rangle_{H_\omega} \in (0,1)$$

indeed if for example $\lambda = 0$ we would have $P\Omega_\omega = 0$ but then $P\varphi_\omega(a)\Omega_\omega = 0$ and by ciclicity of Ω_ω and continuity of P we would deduce that $Pw = 0$ for any $w \in H_\omega$ which is ruled out by non-triviality of P. Similarly $\lambda = 1$ is also ruled out by an analogous argument. Now let

$$\omega_1(a) := \frac{\langle P\Omega_\omega, \varphi_\omega(a)P\Omega_\omega\rangle_{H_\omega}}{\langle P\Omega_\omega, P\Omega_\omega\rangle_{H_\omega}}, \qquad \omega_2(a) := \frac{\langle Q\Omega_\omega, \varphi_\omega(a)Q\Omega_\omega\rangle_{H_\omega}}{\langle Q\Omega_\omega, Q\Omega_\omega\rangle_{H_\omega}},$$

and observe that ω_1, ω_2 are states on $\mathscr{A}$ and that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. If $\omega_1 = \omega_2$ then $\omega = \omega_1 = \omega_2$ and this cannot happen since then

$$\langle \Omega_\omega, \varphi_\omega(a)\Omega_\omega\rangle_{H_\omega} = \frac{\langle P\Omega_\omega, \varphi_\omega(a)\Omega_\omega\rangle_{H_\omega}}{\langle P\Omega_\omega, P\Omega_\omega\rangle_{H_\omega}}, \qquad a \in \mathscr{A}$$

but then $\varphi_\omega(a)\Omega_\omega$ approximate any vector $\psi \in QH_\omega$ but then this implies

$$\langle \Omega_\omega, \psi\rangle_{H_\omega} = \frac{\langle P\Omega_\omega, \psi\rangle_{H_\omega}}{\langle P\Omega_\omega, P\Omega_\omega\rangle_{H_\omega}} = 0,$$

which in turn implies that $Q\Omega_\omega = 0$ but this is a contradiction with $\langle P\Omega_\omega, P\Omega_\omega\rangle_{H_\omega} < 1$. This implies that the state is not extremal, i.e. no pure.

Let's prove the converse, assume that the state ω is not pure, i.e. there exists $\lambda \in (0,1)$ and states $\omega_1 \neq \omega_2$ such that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. This implies that ω_1 is dominated by ω in the sense that if $a \geqslant 0$ we have

$$\omega(a) = \lambda\omega_1(a) + \underbrace{(1-\lambda)\omega_2(a)}_{\geqslant 0} \geqslant \lambda\omega_1(a).$$

So the Hermitian form $B(a,b) \mapsto \omega_1(a^* b)$ on $\mathscr{A}$ satisfies $(\overline{B(a,b)} = \overline{\omega_1(a^* b)} = \omega_1(b^* a) = B(b,a))$

$$B(a,a) \leqslant \frac{1}{\lambda}\omega(a^* a) = \frac{1}{\lambda}\langle a, a\rangle_{H_\omega}.$$

In particular $B(a,b)$ is well defined on $\mathscr{A}\backslash\mathscr{N}_\omega$ with $\mathscr{N}_\omega = \{a\in\mathscr{A}: \langle a,a\rangle_{H_\omega}=0\}$ as a consequence it defines a bounded self-adjoint operator $X: H_\omega \to H_\omega$ such that

$$B(a,b) = \langle a, Xb\rangle_{H_\omega}, \qquad a,b\in\mathscr{A}.$$

(exercise) Now observe that $B(a,cb)=\omega_1(a^* cb)=\omega_1((c^*a)^* b)=B(c^*a,b)$, as a consequence

$$\langle a, X\varphi_\omega(c)\, b\rangle_{H_\omega} = B(a,cb) = B(c^*a,b) = \langle \varphi_\omega(c^*)a, Xb\rangle_{H_\omega} = \langle a, \varphi_\omega(c)Xb\rangle_{H_\omega}, \qquad a,b\in\mathscr{A}$$

from which we conclude that $X\varphi_\omega(c)=\varphi_\omega(c)X$ using the density of $\mathscr{A}$ in H_ω. This holds for any $c\in\mathscr{A}$ therefore we conclude that $X\in\varphi(\mathscr{A})'$. Now X is a non-trivial self-adjoint operator so the representation is not irreducible. □

A pure state represent a situation which cannot be reduced to "simpler ones". If a state is not pure then one can imagine that is obtained probabilistically by sampling one among is pure components with certain probabilities.

Example $\omega = \lambda\,\omega_1 + (1-\lambda)\omega_2$ represents the situation where with probability λ_1 the system is in the state ω_1 and with probability $1-\lambda$ it is in the state ω_2.

Corollary 5. *If $\mathscr{A}$ is Abelian the a state is pure iff it is multiplicative.*

Proof. Let ω be a pure state, then the representation φ_ω is irreducible but it is also Abelian $\varphi_\omega(\mathscr{A})\subseteq\varphi(\mathscr{A})'=\mathbb{C}$, so it is a one-dimensional representation and $H_\omega=\mathbb{C}$ is also a one-dimensional Hilbert space. Therefore

$$\omega(ab) = \langle\Omega_\omega, \varphi_\omega(a)\varphi_\omega(b)\Omega_\omega\rangle_{H_\omega} = \langle\Omega_\omega, \varphi_\omega(a)\Omega_\omega\rangle_{H_\omega}\langle\Omega_\omega, \varphi_\omega(b)\Omega_\omega\rangle_{H_\omega} = \omega(a)\omega(b)$$

so ω is multiplicative. On the hand if ω is multiplicative the $\omega(a^*b)=\omega(a^*)\omega(b)=\overline{\omega(a)}\omega(b)$ so $\varphi_\omega(a)=\omega(a)$ is the GNS representation resulting from it and is one dimensional, therefore irreducible. □

So in the commutative case, the pure state are the elements of the Gelfand spectrum $\Sigma(\mathscr{A})$ and any element of $\mathscr{A}$ can be seen as a continuous complext function on $\Sigma(\mathscr{A})$. A pure state is just evaluation in a point for these functions $\omega(f)=f(\omega)$, i.e. a Dirac measure and a impure state is the limit of convex combintations of such "delta measures". So in particular any state ω can be written as an *average*

$$\omega(f) = \int_{\sigma(\mathscr{A})}\hat{f}(\rho)\mu(\mathrm{d}\rho)$$

for some measure $\mu\in\Pi(\Sigma(\mathscr{A}))$.

So the commutative situation corresponds to standard probability theory and measuraments are "incertain" just because we do not know the pure state that represent the system but maybe only a probability distribution over them.

Note that on pure states ω we have $\Delta_\omega(f)=\omega(f^2)-\omega(f)^2=0$. So they represent the more precise determination of the state of a system. This of course if the algebra is Abelian.