Lecture 8 – Wed May 12th 2020 – 8:15 via Zoom – M. Gubinelli

Last lecture: pure states, irreducible representations, mixed states, probabilistic interpretation.

Our setting (i.e. algebras of observables + states) contains standard probability theory, at least in the case where the algebra is commutative.

Let's agree that an observable is a self-adjoint element of the algebra.

However in general any observable $a \in \mathcal{A}$ define a commutative (C^*-)algebra $C^*(a)$ and therefore any state define a probability measure on $\Sigma(C^*(a)) = \sigma(a) \subseteq \mathbb{R}$.

This can be generalised to a set of commuting observables $(a_1, \ldots, a_n)$ which give the Abelian algebra $C^*(a_1, \ldots, a_n)$ and a measure on $\Sigma(C^*(a_1, \ldots, a_n))$, the set of the pure states (i.e. the multiplicative states) are uniquely labeled by n reals numbers $\{(\omega(a_1), \ldots, \omega(a_n)) : \omega \in \Sigma(C^*(a_1, \ldots, a_n))\} \subseteq \mathbb{R}^n$ So we can identify $\Sigma(C^*(a_1, \ldots, a_n)) = \sigma(a_1) \times \cdots \times \sigma(a_n) \subseteq \mathbb{R}^n$ and any state on this algebra as a probability measure on $\mathbb{R}^n$ with a support on that set.

However in general irreducible representations are not one dimensional if the algebra is non-commutative and they do not corresponds to multiplicative functionals, nor to a probabilistic situation.

Purification. We also saw last time that if a state ω dominates another, e.g. ω_1 (that is if $\omega_1(a) \leqslant C\omega(a)$ for any $a \geqslant 0$) then there exists a non-trivial self-adjoint operator in $\varphi_\omega(\mathcal{A})'$ and therefore there exists also an orthogonal projection $P \in \varphi_\omega(\mathcal{A})'$ and using it is not-difficult to see that H_ω splits into a direct sum $H_\omega = V \oplus W$ and that φ_ω restricts leaves these subspaces invariant and restricts to a sub-representation of $\mathcal{A}$, so we have $\varphi_\omega = \varphi^{(1)} \otimes \varphi^{(2)}$.

Given a representation φ on H we can construct a whole family of states associated to it, called its *folium* they are of the form, for example of a *state vector*

$$\omega^\psi(a) = \langle \psi, \varphi(a)\psi \rangle$$

where ψ is a unit vector in H. Or mixures of state vectors $\psi_1, \ldots, \psi_n$ with weights $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and

$$\omega(a) = \sum_i \lambda_i \langle \psi_i, \varphi(a)\psi_i \rangle = \mathrm{Tr}\left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \varphi(a)\right)$$

where $|\psi_i\rangle\langle\psi_i|$ denotes the rank-1 operator on H given by $|\psi_i\rangle\langle\psi_i|\varphi = \psi_i \langle \psi_i, \varphi\rangle$ for any $\varphi \in H$. More generally we can replace $\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ by any trace class, positive operator $\rho \in \mathscr{L}(H)$. This operator is usually called a density matrix. Recall that Tr is defined on $\mathscr{L}_1(H)$ by

$$\mathrm{Tr}(A) = \sum_n \langle e_n, A e_n \rangle, \qquad A \in \mathscr{L}_1(H) = \left\{ A \in \mathscr{L}(H) : \sum_n |\langle e_n, A e_n \rangle| < \infty \right\}$$

(the definition does not depend on the basis). So a general element of the folium of φ is given by a density matrix ρ

$$\omega^\rho(a) = \mathrm{Tr}(\rho \varphi(a)).$$

Note that ω^ρ is a vector state for its own GNS representation φ_{ω^ρ}, i.e.

$$\omega^\rho(a) = \langle \Omega_{\omega^\rho}, \varphi_{\omega^\rho}(a) \Omega_{\omega^\rho} \rangle_{H_{\omega^\rho}}.$$

Corollary 1. *Any vector state of an irreducible representation is pure.*

I will not prove the following two interesting results.

Theorem 2. *The folium of a representation and the set of vector states of a representation are norm closed subsets in the space of all states $\mathscr{S}$.*

Theorem 3. (Fell) *The folium of a faithful representation is weakly-$*$ dense in the set of all states.*

Remark 4. From a physical point of view we can only do a finite amount of experiments (and with finite precision), which means that we can only identify a weak-$*$ neighborhood of set of all possible states of the system, i.e. a subset of the form

$$\{\omega \in \mathscr{S} : |\omega(a_i) - v_i| \leqslant \varepsilon_i \text{ for all } i = 1, \ldots, n\}$$

where $(a_i)_i$ are observables and $\varepsilon_i > 0$ and $v_i \in \mathbb{R}$. So any faithful representation is as good to be used to approximate a realistic situation. However for mathematical purposes sometimes is useful to single out specific representations which have additional properties.

1 The quantum world

As we saw, it the commutative setting we are able to have states (the pure states) which assign precise values to all observables and where the only source of variance is then described by a probabilistic model. That's something not possible anymore when dealing with microscopic phenomena. This has been discovered at the beginning of the 1900 in various situations and experiments:

- Stern–Gerlach experiment show that the magnetic moment of the electron $M = (M_x, M_y, M_z)$ is quantized (so does not corresponds to the state space which we expect from a vector in $\mathbb{S}^2$) and moreover it seems not to agree with probabilistic reasoning.

- Black-body radiation. The thermodynamical analysis of a particular situation (Plack) at very low temperatures (i.e. $\sim 0^\circ\mathrm{K} \approx 273^\circ C$) pointed out (Einstein) that the degrees of freedom (i.e. different possible states) in the electromagnetic radiation field (light) has to be discrete and not continuous. I.e. light is composed by discrete entities, i.e. photons. That is somehow the set of different possible (pure) states is discrete and not continuous. Planck's constant

$$h = 6.62607004 \times 10^{-34} m^2 \mathrm{kg} / s.$$

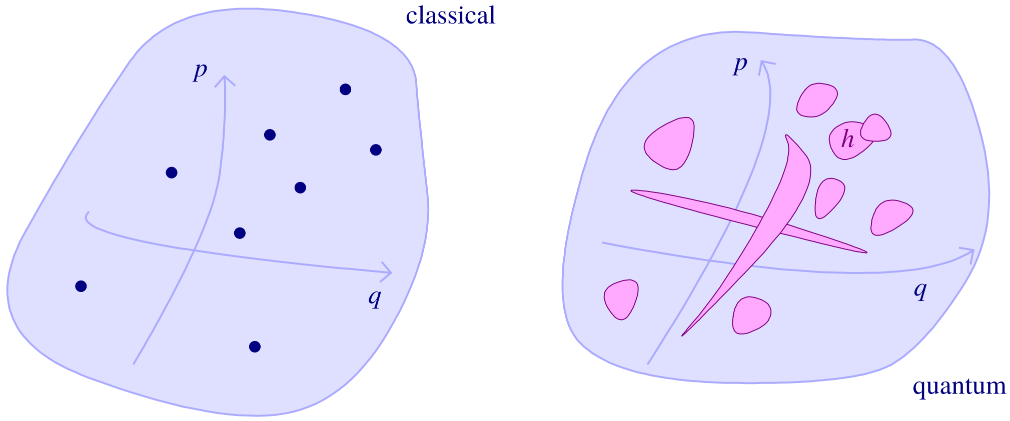
- Heisenberg's analysis of a quantum particle shows that when you try to measure the position and the speed of a particle you get in trouble. In the sense that measurements of position will disturb the velocity of the particle and vice-versa and one should make the hypothesis that both position q and momentum $p = mv$ (i.e. mass times velocity) cannot be determined in any conceivable state ω with arbitrary precision, i.e.

$$\Delta_\omega(q) \Delta_\omega(p) \geqslant \frac{\hbar}{2} \tag{1}$$

  This is Heisenberg's indetermination principle. It somehow implies that the states of a particle cannot be labelled by position and momentum variables, i.e. we need to forbid states which have precise values of position *and* momentum. Note that if (q, p) were forming a commutative algebra then you will have such states like $\delta_{\alpha,\beta}(\mathrm{d}q, \mathrm{d}p)$ which give precise value to $p = \alpha$ and $q = \beta$.

The set of all (elementary) states of a quantum system cannot be put in direct correspondence with the possible values of all the observables. And in particular it is suggested that the set of elementary states is discrete and not continuous.

In classical mechanics the state of a particle is described by a point in the phase space (q, p) of positions and momenta. Any point is possible and any two very nearby points are conceptually distinct. But the existence of the *elementary quantum* h suggests that in every small volume element of size $\delta q\,\delta p \approx h$ there is only one possible quantum state for a particle.



One could put in question the mathematical framework (i.e. restricting states or observables), but actually our setting contains a way out because allows us to introduce *non-commutative* algebras.

This was the conclusion of Heisenberg and he created matrix mechanics, while somehow Schrödinger constructed a different model for the states (i.e. wave-functions constrained by PDEs) and he created wave mechanics. Dirac later showed that the two are equivalent descriptions.

2 The quantum particle

We want now to construct a physical system (observables+states) that encodes Heisenbergs indetermination principle for the position q and momentum p of a particle. So the C^*-algebra of observables $\mathscr{A}$ should contain the C^*-algebra $\mathscr{Q}$ of all the bounded functions $f(q)$ of q and the C^*-algebra $\mathscr{P}$ of all the bounded functions $g(p)$ of p but I need to rule out that q, p commutes otherwise I violate Heisenberg principle unless I restrict the set of states. But restricting the set of states is mode difficult than dealing with a non-commutative algebra because we have more structure on $\mathscr{A}$ than on $\mathscr{S}$.

So we postulate that $[a, b] \neq 0$ at least for some $a \in \mathscr{Q}$ and $b \in \mathscr{P}$ and we let $\mathscr{A}$ to be the smallest C^* algebra containing the abelian subalgebras $\mathscr{Q}$, $\mathscr{P}$. In order for this to describe a single degree of freedom we require that $\Sigma(\mathscr{Q}) \approx \mathbb{R}$ and $\Sigma(\mathscr{P}) \approx \mathbb{R}$.

Next week we will specify completely the structure of this algebra and study its representations.