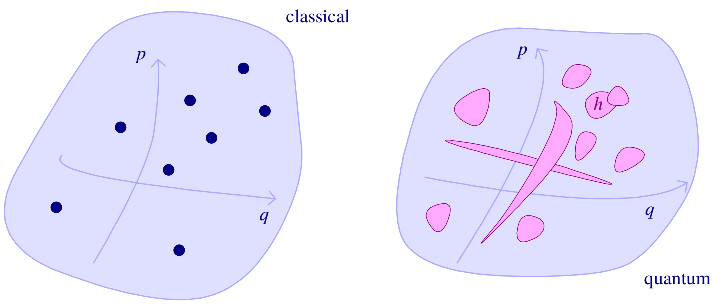
Lecture 9 – May 19th 2020 – 14:15 via Zoom – M. Gubinelli

1 The quantum particle



We want now to construct a physical system (observables+states) that encodes Heisenberg's indetermination principle

$$\Delta_{\omega}(q)\Delta_{\omega}(p) \geqslant \frac{\hbar}{2} \tag{1}$$

for the position q and momentum p of a particle and other experimental observations.

Today we want to explore how non-commutativity is related to the indetermination principle (1) and also to the notion of “complementarity”. Complementary observables are somehow observables which do not allow simultaneous measurement, that is if we are able to have states in which one of the is completely detemined, then the other has to be completely “undetermined”. Think about the Stern-Gerlach experiment and the measurement of the magnetic moment in two orthogonal directions.

For discrete observables q,p Heisenberg's relations do not work very well since there are states where $\Delta_{\omega}(q) = 0$. Also if we change way to measure q, i.e. we measure $f(q)$ for continuous injective $f: \mathbb{R} \to \mathbb{R}$, then I should have the “same information” but somehow the dispertion is different and I can imagine to make dispersion arbitrarily different.

Anyway let us see what we can get from (1). Observe that if $a, b \in \mathscr{A}$ and self-adjoint then $(a + i\lambda b)^*(a + i\lambda b) \geqslant 0$ for any $\lambda \in \mathbb{R}$ and if ω is a state we have

$$0 \leqslant \omega((a + i\lambda b)^*(a + i\lambda b)) = \omega(a^2) + \lambda^2\omega(b^2) + i\lambda\omega(ab - ba),$$

therefore we need to have, letting $[a, b] = ab - ba$,

$$|\omega(i[a, b])| \leqslant 2(\omega(a^2))^{1/2}(\omega(b^2))^{1/2}.$$

So in any C^* algebra we have the (Schrödinger–Robertons) relation

$$\Delta_{\omega}(a)\Delta_{\omega}(b) \geqslant \frac{1}{2}|\omega(i[a, b])|.$$

So if we want to implement Heisenberg's principle for a pair of complementary observables q, p a way is to require that $i[p, q]$ is constant element of $\mathscr{A}$ and therefore

$$[q, p] = i\hbar, \tag{2}$$

These are called canonical commutation relations Heisenberg's matrix mechanics consist in a model where q, p are matrices satisfying the above relation. First problem: these cannot be finite dimensional matrices, indeed if they were we could take the trace over the vector space $\mathbb{C}^n$ they acts on and get

$$\mathrm{Tr}([q,p]) = \sum_n \langle e_n, [q,p] e_n \rangle = 0, \qquad \mathrm{Tr}(i\hbar) = i\hbar n \ldots$$

not very nice. Moreover they cannot implemented even in an abstract C^* algebra, indeed if $q, p \in \mathcal{A}_{sa}$ then

$$[q^n, p] = i\hbar n q^{n-1}$$

and therefore by the C^* condition

$$n\hbar \|q\|^{n-1} = n\hbar \|q^{n-1}\| = \|i\hbar n q^{n-1}\| = \|[q^n, p]\| \leqslant 2\|p\| \, \|q\|^n$$

which implies $\|p\| \, \|q\| \geqslant n\hbar/2$ if $\|q\| \neq 0$. This is true for any n and so either $\|p\|$ or $\|q\|$ has to be infinite.

This somehow is to be expected because "the position" is not really a bounded observable. We cannot really talk about the position of the particle as an element of a C^*-algebra but it is ok if the think to any bounded function of q and an element of the C^* algebra. So we need to avoid to talk about q and talk instead of a C^* algebra $\mathcal{Q}$ which plays the role of the algebra of functions of the position, that is has to be a commutative C^* algebra without unit (in order to allow for non-compact spectrum).

At this point is not clear how to single out an algebra of observables which satisfies something like the indetermination principle.

2 Non-commutativity and probability

To start simpler we consider first system which possess "finitely many" pure states. Think about the two states in the Stern–Gerlach experiment. Let us assume we have two observables a, b which generates $\mathcal{A}$ and such that $\sigma(a), \sigma(b)$ are finite. Is not difficult to show that I can construct projections $(\pi_k^a)_k \subseteq C^*(a)$ such that $\pi_k^a \pi_\ell^a = \delta_{k,\ell} \pi_k^a$, $\sum_k \pi_k^a = 1$ and $\pi_k^a f(a) = f(a_k) \pi_k^a$ where $\{a_k : k = 1, \ldots, n^a\} = \sigma(a)$ is an enumeration of the spectrum of a. Let us also assume that $C^*(a)$ and $C^*(b)$ are maximally abelian subalgebras in $\mathcal{A}$. Now observe that $\sum_k \pi_k^a \pi_\ell^b \pi_k^a$ commutes with any $h \in C^*(a)$ and therefore $\sum_k \pi_k^a \pi_\ell^b \pi_k^a \in C^*(a)$ by maximal abelianity, and then there exists numbers $(p_{\ell,k}^{b|a})_{k,\ell}$ such that

$$\sum_k \pi_k^a \pi_\ell^b \pi_k^a = \sum_k p_{\ell,k}^{b|a} \pi_k^a, \qquad \ell = 1, \ldots, n^b.$$

Observe that by positivity of the various projectors π^a, π^b we must have $p_{\ell,k}^{b|a} \geqslant 0$ and moreover $\sum_\ell p_{\ell,k}^{b|a} = 1$.

Therefore we have a set of probabilities $(p_{\ell,k}^{b|a})_{k,\ell}$ which are generated intrinsically by the non-commutativity of the algebra, even before we consider the states on that algebra.

For any state ω we can construct a new state $\omega^a(h) = \sum_k \omega(\pi_k^a h \pi_k^a)$ and now observe that

$$\omega^a(f(a)) = \omega(f(a)), \qquad \omega^a(f(b)) = \sum_{k,\ell} f(b_\ell) \omega(\pi_k^a \pi_\ell^b \pi_k^a) = \sum_{\ell,k} f(b_\ell) p_{\ell,k}^{b|a} \omega(\pi_k^a)$$

so $\omega^a(\pi_\ell^b) = \sum_k p_{\ell,k}^{b|a} \omega(\pi_k^a)$. So $p_{\ell,k}^{b|a}$ can be interpreted as the conditional probability to observe $b = b_\ell$ given we have observed $a = a_k$. Note also that the state ω^a is never pure if a, b do not commute.

3 Complementary observables in finite quantum system

Consider still systems with finitely many pure states. All the observables have to take only finitely many values. So we can assume that they have all the same spectrum with n points and to be given by

$$\Gamma = \{\gamma_k = e^{2\pi i k/n}\}_{k=0,\ldots,n-1}.$$

We want to construct an algebra of two non-commuting observables u, v where both have the same spectrum (as above) and they are complementary. A natural way to understand complementarity using the matrix $p^{v|u}_{\ell,k}$ is to impose that $p^{v|u}_{\ell,k} = 1/n$ for any k, ℓ. The question is now, can do it? This would provide a system where the two observable u, v are complementary. Indeed, if we can impose this condition we would be able to construct states under which u has a definite value and v is uniform in Γ.

By GN theorem we can look for construction of this algebra as a family of operators in an separable Hilbert space. Is clear we need at least a space of dimensions n otherwise we cannot accomodate the n different eigenvalues Γ. By abuse of language u, v the representatives of u, v in the space $\mathscr{L}(\mathbb{C}^n)$. Note that u, v have to be unitary operators. Let $(\varphi_k)_k$ be the eigenvectors of u, i.e. $u\varphi_k = \gamma_k \varphi_k$ and then take $v\varphi_k = \varphi_{k+1}$ with $k+1$ understood modulus n. Now observe that $uv\varphi_k = u\varphi_{k+1} = \gamma_{k+1}\varphi_{k+1} = \gamma_{k+1} v\varphi_k = (\gamma_{k+1}/\gamma_k) vu\varphi_k$ for any $k = 0, \ldots, n-1$ so

$$uv = e^{2\pi i/n} vu.$$

Observe also that $u^n = v^n = 1$. In particular

$$0 = (\gamma_k^{-1} u)^n - 1 = (\gamma_k^{-1} u - 1) \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$$

and from this we deduce that $\pi^u_k = \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$ satisfies $u\pi^u_k = \gamma_k \pi^u_k$ so π^u_k is the orthogonal projection on the span of φ_k.