Markov Processes – Problem Sheet 1.

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Solutions will be collected Thursday November 2nd during the lecture. At most in groups of 2.

Exercise 1. Show that standard Brownian motion and the Poisson process with intensity λ are both (homogeneous) Markov processes with Feller transition kernels.

Exercise 2. Let $(N_t)_t$ be a Poisson process with intensity $\lambda > 0$.

a) Let $(Y_n)_{n \geqslant 1}$ an iid family of real r.v. with $\nu = \mathrm{Law}(Y_n) \in \mathcal{P}(\mathbb{R})$. For any $x \in E$ consider the cadlag process

$$X_t = x + \sum_{n=1}^{N_t} Y_n.$$

Show that this is a Markov process, compute its transition kernel and show that is Feller.

b) Let now $(Z_n)_{n \geqslant 0}$ be a Markov chain on the state space $(E, \mathcal{E})$ independent of $(N_t)_{t \geqslant 0}$ and consider the process $X_t = Z_{N_t}$. Show that $(X_t)_t$ is a Markov process on $(E, \mathcal{E})$ and compute its transition kernel. Show directly that this kernel statisfy the Chapman–Kolmogorov equation.

Exercise 3. Let $(X_t)_t$ be an inhomogeneous Markov process with transition kernel $(P_{s,t})_{s,t}$. Show that the process $\hat{X}_t = (t, X_t)$ is a Markov process with homogeneous transition kernel $(\hat{P}_t)_t$ and give the expression of this kernel as a function of the original kernel.

Exercise 4. Prove the following properties of stopping times on the standard setup (càdlàg paths, right continuous filtration)

a) for any open $G \subset E$ the r.v. $\tau_G = \inf \{t \geqslant 0: X_t \in G\}$ is a stopping time;

b) if τ, σ are stopping times, then $\tau \wedge \sigma$, $\tau \vee \sigma$ and $\tau + \sigma$ are stopping times;

c) if $\tau \leqslant \sigma$ then $\mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$, if $\tau_n \downarrow \tau$ then $\mathcal{F}_\tau = \cap_n \mathcal{F}_{\tau_n}$;

Exercise 5. Prove the strong Markov property for a Feller process. Namely prove that for any stopping time τ and bounded measurable function $F: \mathbb{R}_{\geqslant 0} \times \Omega \to \mathbb{R}$ we have

$$\mathbb{E}_x[F_\tau \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[F_\tau] \qquad \text{on } \{\tau < \infty\},\ \mathbb{P}_x\text{–a.s. for any } x \in E.$$

Recall the specific interpretation of this notations from the course notes. You may want to follow this strategy: first prove it for discrete stopping times τ; then for arbitrary τ and functions $Y(t, \omega) = f(t) \prod_i f_i(\omega(t_i))$ where $\{t_i\}_i$ is a finite collections of times and f, f_i are bounded continuous functions; conclude by a monotone class argument.

Exercise 6. Let $(\mathbb{P}_x)_x$ be a Brownian motion and let $\tau_a = \inf \{t > 0: X_t = t + a\}$ for $a > 0$. Assume that $\mathbb{P}_x(\tau_a < \infty) = 1$ for all $a > 0$. Use the strong Markov property for Brownian motion to prove that

$$\mathbb{P}_0(\tau_{a+b} < \infty | \tau_a < \infty) = \mathbb{P}_0(\tau_b < \infty).$$

And then deduce from this that the random variable $\sup_{t \geqslant 0} [X_t - t]$ has exponential distribution with some parameter λ (you can leave it undertermined).