UNIVERSITÄT BONN IAM

Markov Processes – Problem Sheet 10.

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Solutions will be collected Tuesday January 16th during the lecture. At most in groups of 2.

Exercise 1. (Gibbs sampler for the Ising model). [5pts] Consider a finite graph (V, F) with n vertices of maximal degree Δ. The Ising model with inverse temperature $\beta \geqslant 0$ is the probability measure μ_β on $\{-1,1\}^V$ with mass function

$$\mu_\beta(\eta) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{x\sim y} \eta(x)\eta(y) \right),$$

where $Z(\beta)$ is the normalization constant and $x \sim y \Leftrightarrow \{x, y\} \in F$.

a) Show that given $\eta(y)$ for $y \neq x$, $\eta(x) = \pm 1$ with probability $(1 \pm \tanh(\beta m(x, \eta)))/2$, where $m(x, \eta) := \sum_{y\sim x} \eta(y)$ is the local magnetization in the neighbourhood of x. Hence determine the transition kernel π for the Gibbs sampler with equilibrium μ_β.

b) Prove that for any $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu\pi^t, \mu\beta) \leqslant \alpha(n, \beta, \Delta)^t \mathcal{W}^1(\nu, \mu_\beta) \leqslant \exp\left(-\frac{t}{n}(1 - \Delta \tanh(\beta)) \right) \mathcal{W}^1(\nu, \mu_\beta),$$

where $\alpha(n, \beta, \Delta) = 1 - (1 - \Delta \tanh(\beta))/n$, and $\mathcal{W}^1$ is the transportation metric based on the Hamming distance on $\{-1, 1\}^V$. Conclude that for $\Delta \tanh(\beta) < 1$, the Gibbs sampler is geometrically ergodic with a rate of order $\Omega(1/n)$. Hint: You may use the inequality

$$|\tanh(y + \beta) - \tanh(y - \beta)| \leqslant 2 \tanh(\beta) \text{ for any } \beta \geqslant 0 \text{ and } y \in \mathbb{R}.$$

c) The *mean-field Ising model* with parameter $\alpha \geq 0$ is the Ising model on the complete graph over $V = \{1, \ldots, n\}$ with inverse temperature $\beta = \alpha/n$. Show that for $\alpha < 1$, the ε-mixing time for the Gibbs sampler on the mean field Ising model is of order $O(n \log n)$ for any $\varepsilon \in (0, 1)$.

Exercise 2. (Bounds for ergodic averages in the non-stationary case) [5pts] Let (X_n, P_x) be a

Markov chain with transition kernel π and invariant probability measure μ, and let

$$A_{b,n} f = \frac{1}{n} \sum_{i=b}^{b+n-1} f(X_i).$$

Assume that there are a distance d on the state space E, and constants $\alpha \in (0, 1)$ and $\bar{\sigma} \in \mathbb{R}_+$ such that

(A1). $\mathcal{W}_d^1(\nu\pi, \tilde{\nu}\pi) \leqslant \alpha \mathcal{W}_d^1(\nu, \tilde{\nu})$, for all $\nu, \tilde{\nu} \in \mathcal{P}(E)$, and

(A2). $\mathrm{Var}_{\pi(x,\cdot)}(f) \leqslant \bar{\sigma}^2 \|f\|_{\mathrm{Lip}(d)}^2$ for all $x \in E$, $f: E \to \mathbb{R}$ Lipschitz.

Prove that under these assumptions the following bounds hold for any $b, n, k \geqslant 0$, $x \in E$, and for any Lipschitz continuous function $f: E \to \mathbb{R}$:

a) $\mathrm{Var}_x[f(X_n)] \leqslant \sum_{k=0}^{n-1} \alpha^{2k} \bar{\sigma}^2 \|f\|_{\mathrm{Lip}(d)}^2$.

b) $|\mathrm{Cov}_x[f(X_n), f(X_{n+k})]| = |\mathrm{Cov}_x[f(X_n), \pi^k f(X_n)]| \leqslant \alpha^k (1-\alpha^2)^{-1} \bar{\sigma}^2 \|f\|_{\mathrm{Lip}(d)}^2$.

c) $\mathrm{Var}_x[A_{b,n} f] \leqslant \frac{1}{n} \frac{\bar{\sigma}^2 \|f\|_{\mathrm{Lip}(d)}^2}{(1-\alpha)^2}$.

d) $|\mathbb{E}_x[A_{b,n} f] - \mu(f)| \leqslant \frac{1}{n} \frac{\alpha^b}{1-\alpha} \int d(x,y) \mu(\mathrm{d}y) \|f\|_{\mathrm{Lip}(d)}$.

e) $\mathbb{E}_x[|A_{b,n} f - \mu(f)|^2] \leqslant \frac{1}{n} \frac{1}{(1-\alpha)^2} \left(\bar{\sigma}^2 + \frac{\alpha^{2b}}{n} (\int d(x,y) \mu(\mathrm{d}y))^2 \right) \|f\|_{\mathrm{Lip}(d)}^2$.

Exercise 3. (Equivalent descriptions for weighted total variation norms) [5pts] Let $V: E \to (0, \infty)$ be a measurable function, and let $d_V(x,y) := (V(x) + V(y)) 1_{x \neq y}$. Show that the following identities hold for probability measures μ, ν on $(E, \mathcal{E})$:

$$\begin{aligned}
\|\mu - \nu\|_{\mathrm{TV}} &= \left\| \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} - \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \right\|_{L^1(V \cdot \lambda)} \\
&= \sup \{ |\mu(f) - \nu(f)| : f \in \mathcal{F}(E) \ s.t. \ |f| \leqslant V \} \\
&= \sup \{ |\mu(f) - \nu(f)| : f \in \mathcal{F}(E) \ s.t. \ |f(x) - f(y)| \leqslant d_V(x,y) \ \forall x, y \in E \} \\
&= \inf \{ \mathbb{E}[d_V(X,Y)] : X \sim \mu, Y \sim \nu \}
\end{aligned}$$