Markov Processes – Problem Sheet 11.

Tutorials by Nikolay Barashkov `<s6nibara@uni-bonn.de>`, Robert Crowell `<crowellr@googlemail.com>`.
Solutions will be collected Tuesday January 23rd during the lecture. At most in groups of 2.

Exercise 1. (Infinitesimal characterization of invariant measures). [5pts] Consider a time-homogeneous continuous time Markov chain $X_t = Y_{N_t}$ where (N_t) is a Poisson process with constant intensity $\lambda > 0$, and (Y_n) is an independent Markov chain with transition matrix π on a finite state space E.

a) Show that the transition function is given by

$$p_t(x, y) = \mathbb{P}_x[X_t = y] = \exp(t\mathcal{L})(x, y),$$

where $\mathcal{L} = \lambda(\pi - I)$ and $\exp(t\mathcal{L})$ is the matrix exponential. Hence conclude that $(p_t)_{t \geqslant 0}$ satisfies the forward and backward equation

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = p_t\mathcal{L} = \mathcal{L}p_t \text{ for } t \geqslant 0.$$

b) Prove that a probability measure μ on E is invariant for (p_t) if and only if

$$\sum_{x \in E} \mu(x)\mathcal{L}(x, y) = 0 \qquad \text{for any } y \in E.$$

c) Show that the transition matrices are self-adjoint in $L^2(\mu)$, i.e.,

$$\sum_{x \in E} \mu(x) f(x) p_t g(x) = \sum_{x \in E} \mu(x) p_t f(x) g(x), \qquad \text{for any } t \geqslant 0,\ f, g : E \to \mathbb{R},$$

if and only if the generator $\mathcal{L}$ satisfies the detailed balance condition w.r.t. μ. What does this mean for the process?

Exercise 2. (Simple exclusion process) [5pts] Let $\mathbb{Z}_n^d = \mathbb{Z}^d / (n\mathbb{Z})^d$ denote a discrete d-dimensional torus. The simple exclusion process on $E = \{0, 1\}^{\mathbb{Z}_n^d}$ is the Markov process with generator

$$\mathcal{L}f(\eta) = \frac{1}{2d} \sum_{x \in \mathbb{Z}_n^d} \sum_{y : |x - y| = 1} \mathbb{I}_{\{\eta(x) = 1, \eta(y) = 0\}} (f(\eta^{x,y}) - f(\eta))$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values of x and y. Show that any Bernoulli measure of type

$$\mu_p = \otimes_{x \in \mathbb{Z}_n^d} \nu_p, \qquad \nu_p(1) = p, \quad \nu_p(0) = 1 - p,$$

$p \in [0, 1]$ is invariant. Why does this not contradict the fact that any irreducible Markov process on a finite state space has a unique stationary distribution? *(You may assume the statements of Exercise 1).*

Exercise 3. (IMMIGRATION-DEATH PROCESS) [5pts] Particles in a population die independently with rate $\mu>0$. In addition, immigrants arrive with rate $\lambda>0$. Assume that the population consists initially of one particle.

a) Explain why the population size X_t can be modeled by a birth-death process with rates $b(n)=\lambda$ and $d(n)=n\mu$.

b) Show that the generating function $G(s,t)=\mathbb{E}(s^{X_t})$ is given by

$$G(s,t)=(1+(s-1)e^{-\mu t})\exp\left(\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})\right).$$

c) Deduce the limiting distribution of X_t as $t\to\infty$.

Exercise 4. (A NON-EXPLOSION CRITERION FOR JUMP PROCESSES) [5pts] Suppose that $q_t(x, B)=\lambda_t(x)\,\pi_t(x,\ B)$ where π_t is a probability kernel on $(E,\ \mathcal{E})$ and $\lambda_t: E\to[0,\ \infty)$ is a measurable function. We consider the minimal jump process $((X_t), P_{t_0,x_0})$ with jump times (J_n) and positions (Y_n) defined by the following algorithm:

1. Set $J_0=t_0$ and $Y_0=x_0$.

2. For $n=1,2,..$ do

   i. Sample $E_n\sim\mathrm{Exp}(1)$ independently of $Y_0,\ldots,Y_{n-1}$ and $E_0,\ldots,E_{n-1}$;

   ii. Set $J_n=\inf\left\{t\geqslant J_{n-1}\colon \int_{J_{n-1}}^t\lambda_s(Y_{n-1})\mathrm{d}s\geqslant E_n\right\}$;

   iii. Sample $Y_n|(Y_0,\ldots,Y_{n-1},E_0,\ldots,E_n)\sim\pi_{J_n}(Y_{n-1},\cdot)$.

a) Show that if $\bar{\lambda}:=\sup_{t\geqslant 0}\sup_{x\in E}\lambda_t(x)<\infty$, then the explosion time $\zeta=\sup J_n$ is almost surely infinite.

b) In the time-homogeneous case, given $\sigma(Y_k: k\in\mathbb{Z}_+)$,

$$J_n=\sum_{k=1}^n\frac{E_k}{\lambda(Y_{k-1})}$$

is a sum of conditionally independent exponentially distributed random variables. Conclude that the events $\{Y<\infty\}$ and $\{\sum_{k=0}^\infty(\lambda(Y_k))^{-1}<\infty\}$ coincide almost surely (apply Kolmogorov's 3-series theorem).