Markov Processes – Problem Sheet 2.

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Solutions will be collected Tuesday November 7th during the lecture. At most in groups of 2.

Exercise 1. (Random Recurrences) Prove that for any discrete Markov chain (i.e. on a countable state space E) with one step probability π and starting at $x_0 \in E$ can be realized as the random recurrence

$$X_{n+1} = \Phi(X_n, U_n), \qquad X_0 = x_0,$$

where $(U_n)_n$ is a sequence of i.i.d r.v. uniformly distributed in $[0,1]$ and $\Phi: E \times [0,1] \to E$ is a suitable function.

Exercise 2. (Regeneration) Let $(X_n)_n$ be a irreducible recurrent Markov chain on a countable state space E, realised on the canonical space with shift $(\theta_n)_n$. For $A \subseteq E$ and $T_A^+ := \inf\{n > 0: X_n \in A\}$, $T_A^{n+1} = T_A^+ \circ \theta_{T_A^n}$ the return times to A and for $x \in E$ let $T_x^n = T_{\{x\}}^n$.

a) Fix $x \in E$ and prove that the interarrival times $(\tau_x^n := T_x^n - T_x^{n-1})_{n \geqslant 1}$ at x are i.i.d.

b) Fix $A \subseteq E$ and prove that for any $x \in A$, the process $(Y_n = X_{T_A^n})_{n \geqslant 0}$ under $\mathbb{P}_x$ is a Markov chain on the state space A. Compute its one step probabilities.

Exercise 3. (Reflection principle) Let $(B_t)_t$ be the standard Brownian motion and $M_t = \sup_{s \leqslant t} B_s$ its running maximum. Prove that for any $a > 0$ and $t > 0$ we have

$$\mathbb{P}_0(M_t \geqslant a) = 2\mathbb{P}_0(B_t \geqslant a) = \mathbb{P}_0(|B_t| \geqslant a).$$

In order to do so, note that $\mathbb{P}_0$-a.s. we have $\{M_t \geqslant a\} = \{\tau_a \leqslant t\}$ where $\tau_a = \inf\{t \geqslant 0: B_t = a\}$.

Exercise 4. (Martingales)

a) Let $(M_n)_n$ be a supermartingale such that $\mathbb{E}[M_n]$ is constant. Show that it is a martingale.

b) Let $(Z_n)_n$ be an integrable adapted process. Show that it is a martingale iff for any bounded stopping time T one has $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

(turn the page)

Exercise 5. (ROBBINS-MONROE ALGORITHM) This is a stochastic algorithm to estimate the α-quantile q_α of a given continuous repartition function $F_X(t) := \mathbb{P}(X \leqslant t)$ given any level $\alpha \in (0, 1)$. Recall that $q_\alpha = F(q_\alpha)$ and is the only solution to this equation. Let $(X_n)_{n\geqslant 1}$ an iid sequence with law F and let $(Y_n)_{n\geqslant 0}$ the sequence defined by the recursion

$$Y_{n+1} = Y_n - \gamma_n(\mathbb{I}_{X_{n+1}\leqslant Y_n} - \alpha) \qquad n \geqslant 0$$

where Y_0 is an arbitrary constant. The sequence $(\gamma_n)_{n\geqslant 0}$ is positive, decreasing and such that $\sum \gamma_n^2 < +\infty$ and $\sum \gamma_n = +\infty$. We want to show that $Y_n \to q_\alpha$ a.s.

a) Let $(Z_n = (Y_n - q_\alpha)^2)_{n\geqslant 0}$. Compute $\mathbb{E}[Z_{n+1}|\mathcal{F}_n]$ and show that there exists a decreasing and bounded sequence $(U_n)_{n\geqslant 1}$ such that $W_n := Z_n - U_n$ satisfies

$$0 \leqslant \gamma_n(Y_n - q_\alpha)(F(Y_n) - \alpha) \leqslant W_n - \mathbb{E}[W_{n+1}|\mathcal{F}_n] \tag{1}$$

b) Show that $(W_n)_{n\geqslant 1}$ converges a.s.

c) Show that eq. (1) implies that the series

$$\sum_{n\geqslant 0} \gamma_n(Y_n - q_\alpha)(F(Y_n) - \alpha)$$

converges in L^1 and a.s. and that together with the condition $\sum \gamma_n = +\infty$ this implies that $Y_n \to q_\alpha$ a.s.