Institute for Applied Mathematics – WS17/18
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Markov Processes – Problem Sheet 3.

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Solutions will be collected Tuesday November 14th during the lecture. At most in groups of 2.

Exercise 1. (Feynman–Kac formula) [5 pt] Let $(X_n)_n$ be a Markov chain on $(E,\mathcal{E})$ with generator $\mathcal{L}$. Let $w: E \to \mathbb{R}_+$ a non-negative function:

a) For which function v the process

$$M_n = \exp[-\sum_{k=0}^{n-1} w(X_k)] v(X_n)$$

is a martingale?

b) Let $D \in \mathcal{E}$ such that $T = T_{D^c} < \infty$ $\mathbb{P}_x$-a.s. for all $x \in E$ and let v be a bounded solution of the boundary value problem

$$\mathcal{L}v(x) = (e^{w(x)} - 1)v(x), \qquad x \in D$$
$$v(x) = f(x), \qquad x \in D^c.$$

Show using (a) that

$$v(x) = \mathbb{E}_x\left[\exp\left(-\sum_{k=0}^{T-1} w(X_k)\right) f(X_T)\right].$$

Exercise 2. (Recurrence of simple RW) [5 pt] Use the Forster–Lyapounov criteria to show that the simple RW on $\mathbb{Z}^d$ is recurrent if $d \leqslant 2$ and transient if $d > 2$.

Exercise 3. (Stochastic amplification) [5 pt] Consider the Markov chain $X_{n+1} = A_n X_n + \xi_n$ on $\mathbb{R}$ where $(\xi_n)_n$ is a sequence of iid Gaussian r.v. with mean zero and unit variance and $(A_n)_n$ is a iid sequence of positive and bounded r.v.. Show that for this chain the unit ball is positive recurrent if $\mathbb{E}\log \mathrm{A}_1 < 0$.

Exercise 4. (Max-Weight queue scheduling) [5 pt] Consider N discrete time queues served by a single server. In each queue $i \in \{1,\dots,N\}$, the number of existing customers, the number of customer arrivals, and the available service are denoted by X_n^i, A_n^i, and R_n^i respectively at each discrete time slot $n \geqslant 0$. Arrivals $(A_n^i)_{i,n}$ at each queue $i \in \{1,\dots,N\}$ and at each time $n \geqslant 1$ is assumed to be independent, and it satisfies the following conditions,

$$\mathbb{E}[A_n^i] = \lambda_i, \quad \mathbb{P}(A_n^i = 0) > 0, \qquad \mathbb{E}[(A_n^i)^2] \leqslant C.$$

This implies mean rate of arrival to the system is $\sum_i \lambda_i$. The server is shared by all N queues, and can only serve one customer in any one queue in any time slot n. The customer leaves the queue if served. Hence, the maximum rate of departure from all the queues is unity. We will assume that aggregate mean rate of arrival to the system is smaller than maximum available service to the system, that is $\sum_i \lambda_i < 1$. Let I_n be the queue index served by the server in each time n, then the service available to queue i at time n is $R_n^i = \mathbb{I}_{I_n = i}$. We can write the time evolution for queue i as

$$X_{n+1}^i = \max(X_n^i + A_n^i - R_n^i, 0).$$

At each time n, the selection of queue I_n to be served is called *scheduling*. Consider a scheduling policy based solely on the current state of all queues $X_n = (X_n^1, \ldots, X_n^N)$. For such policies, the aggregate state of the queues form a Markov chain, since each arrival is independent, and queue evolution only depends on the current state. This is a difficult Markov chain to study if the scheduling policy is not random. We are interested in a scheduling policy that would make the Markov chain positive recurrent if $\sum_i \lambda_i < 1$. *Max-weight* scheduling policy selects the longest queue for service at each time n:

$$I_n = \operatorname{argmax}\{X_n^i : i \in \{1, \ldots, N\}\}.$$

Use Forster–Lyapounov criteria to prove that this chain is irreducible and positive recurrent if $\sum_i \lambda_i < 1$.